Topological matings and ray connections

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Abstract

A topological mating is a map defined by gluing together the filled Julia sets of two quadratic polynomials. The identifications are visualized and understood by pinching ray-equivalence classes of the formal mating. For postcritically finite polynomials in non-conjugate limbs of the Mandelbrot set, classical results construct the geometric mating from Thurston theory. Here families of examples are discussed, such that all ray-equivalence classes are uniformly bounded trees. Thus the topological mating is obtained directly in geometrically finite and infinite cases. On the other hand, renormalization provides examples of unbounded cyclic ray connections, such that the topological mating is not defined on a Hausdorff space. Moreover, matings with long ray connections are found algorithmically.

1 Introduction

Starting from two quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, construct the topological mating $P \coprod Q$ by gluing the filled Julia sets $\mathcal{K}_p$ and $\mathcal{K}_q$. If there is a conjugate rational map f, this defines the geometric mating. These maps are understood by starting with the formal mating $g = P \sqcup Q$, which is conjugate to P on the lower half-sphere $|z| < 1$ and to Q on the upper half-sphere $|z| > 1$ of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: ray-equivalence classes consist of external rays of P and Q with complex conjugate angles, together with landing points in $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_q$; collapsing these classes defines the topological mating. In the postcritically finite case, with p and q not in conjugate limbs of $\mathcal{M}$, either g or a modified version $\widetilde{g}$ is combinatorially equivalent and semi-conjugate to a rational map f [45, 7, 12, 19, 43]. So the topological mating exists and f is conjugate to it — it is a geometric mating.

Basic definitions and the geometry of ray-equivalence classes are discussed below. In general there is only a Cantor set of angles at the Hubbard tree $T_{\overline{q}} \subset \mathcal{K}_{\overline{q}}$, whose Hausdorff dimension is less than 1. If an open interval in the complement contains all angles on one side of the arc $[-\alpha_p\, , \alpha_p] \subset \mathcal{K}_p$, ray connections of the formal mating $P \sqcup Q$ are bounded explicitly, and the topological mating exists. This approach was used by Shishikura–Tan in a cubic example [44]; in the quadratic case it generalizes the treatment of $1/4 \coprod 1/4$ by Milnor [33] to large classes of examples. These include the mating of Airplane and Kokopelli, answering a question by Adam Epstein [6]: can the mating be constructed without employing the theorems of Thurston and Rees–Shishikura–Tan? See **Section 3**. Note however, that only the branched cover on the glued Julia sets is constructed here, not a conjugate rational map. On the other hand, the method applies to geometrically infinite parameters as well. Examples of irrational ray connections and an algorithm for finding long ray connections are discussed in addition. In **Section 4**, specific ray connections for polynomials from conjugate limbs are obtained, which are related to renormalization of one polynomial. These ray-equivalence classes accumulate on the Julia set, such that the quotient space is not Hausdorff.

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2 Mating: definitions and basic properties

After recalling basic properties of quadratic polynomials and matings, the geometry of rational and irrational ray-equivalence classes is described, generalizing an observation by Sharland [42].

2.1 Polynomial dynamics and combinatorics

For a quadratic polynomial $f_c(z) = z^2 + c$, the filled Julia set $\mathcal{K}_c$ contains all points z with $f_c^n(z) \not\to \infty$. It is connected, if and only if the critical point $z = 0$ does not escape, and then the parameter c belongs to the Mandelbrot set $\mathcal{M}$ by definition. A **dynamic ray** $\mathcal{R}_c(\theta)$ is the preimage of a straight ray with angle $2\pi\theta$ under the Boettcher conjugation $\Phi_c : \widehat{\mathbb{C}} \setminus \mathcal{K}_c \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. For rational θ, the rays and landing points are periodic or preperiodic under f_c , since $f_c(\mathcal{R}_c(\theta)) = \mathcal{R}_c(2\theta)$. If two or more periodic rays land together, this defines a non-trivial orbit portrait; it exists if and only if the parameter c is at or behind a certain root [37, 32]. There are analogous parameter rays with rational angles $\mathcal{R}_M(\theta)$ landing at roots and Misiurewicz points; the angles of a root are characteristic angles from the orbit portrait. In particular, the k/r-limb and wake of the main cardioid are defined by two parameter rays with r-periodic angles, and for the corresponding parameters c, the fixed point $\alpha_c \in \mathcal{K}_c$ has r branches and external angles permuted with rotation number k/r. Denote landing points by $z = \gamma_c(\theta) \in \partial\mathcal{K}_c$ and $c = \gamma_M(\theta) \in \partial\mathcal{M}$, respectively. f_c is **geometrically finite**, if it is preperiodic, hyperbolic, or parabolic.

2.2 Topological mating and geometric mating

For parameters p, $q \in \mathcal{M}$ with locally connected Julia sets, define the **formal mating** $g = P \sqcup Q$ of the quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ as follows: $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched cover with critical points 0 and ∞, and normalized such that $g(z) = z^2$ for $|z| = 1$. On the lower and upper half-spheres, g is topologically conjugate to P and Q by homeomorphisms φ_0 and φ_∞ , respectively. An **external ray** $\mathcal{R}(\theta)$ of g is the union of $\varphi_0(\mathcal{R}_p(\theta))$ and $\varphi_\infty(\mathcal{R}_q(-\theta))$ plus a point on the equator; each ray connects a point in $\varphi_0(\mathcal{K}_p)$ to a point in $\varphi_\infty(\mathcal{K}_q)$. A **ray-equivalence class** is a maximal connected set consisting of rays and landing points. Collapsing all classes to points may define a Hausdorff space homeomorphic to the sphere; then the map corresponding to g is a branched cover again [35], which defines the **topological mating** $P \coprod Q$ up to conjugation. By the identifications, periods may be reduced and different orbits may meet. We are interested in a rational map f conjugate to the topological mating, and we shall speak of "the" geometric mating when the following normalization is used. Note however, that uniqueness is not obvious when the polynomials are not geometrically finite, in particular if the rational map has an invariant line field.

Definition 2.1 (Normalization of the geometric mating)
Suppose the topological mating $P \coprod Q$ is topologically conjugate to a quadratic rational map F, and the conjugation ψ is conformal in the interior of the filled Julia sets. Then the geometric mating exists and it is Möbius conjugate to F.

*The **geometric mating** $f \cong P \coprod Q$ is normalized such that ψ maps the critical point of P to 0, the critical point of Q to ∞, and the common β-fixed point to 1. If the latter condition is dropped, then f is **linear conjugate** to the geometric mating, and we shall write $f \simeq P \coprod Q$.*

Sometimes it is convenient to write $p \coprod q$ or $\theta_p \coprod \theta_q$ for $P \coprod Q$; here a periodic angle is understood to define a center, not a root. In the postcritically finite case, the geometric mating is constructed using Thurston theory as follows:

Theorem 2.2 (Rees–Shishikura–Tan)
Suppose P and Q are postcritically finite quadratic polynomials, not from conjugate limbs of the Mandelbrot set. Then the geometric mating $f \cong P \coprod Q$ exists.

Idea of the **proof:** The formal mating $g = P \sqcup Q$ is a postcritically finite branched cover, a Thurston map. So it is combinatorially equivalent to a rational map, if and only if it is unobstructed, excluding type (2, 2, 2, 2) here [22]. According to Rees–Shishikura–Tan, all obstructions contain Lévy cycles converging to ray-equivalence classes under iterated pullback [45]. See the example in Figure 3 of [24]. In the case of non-conjugate limbs, these obstructions are removed by collapsing postcritical ray-equivalence trees, which defines an unobstructed essential mating $\widetilde{g}$. Now the Thurston Theorem [7, 12, 19] produces a rational map f equivalent to g or $\widetilde{g}$, respectively, unique up to normalization. Pulling back a suitable equivalence, a semi-conjugation from g to f is obtained [43], which collapses all ray-equivalence classes to points. So f is conjugate to the topological mating $P \coprod Q$. ∎

Conjecture 2.3 (Quadratic mating)
For quadratic polynomials P and Q with locally connected Julia sets, the geometric mating exists, unless p and q are in conjugate limbs of the Mandelbrot set.

Originally, it was expected that mating depends continuously on the polynomials [30]; various counterexamples by Adam Epstein [15, 6] are discussed in [21], and a simple new counterexample is given. — The geometric mating of quadratic polynomials is known to exist in the following situations:

- In the postcritically finite case, Conjecture 2.3 was proved in [45, 43, 22], cf. Theorem 2.2. Here the geometric mating exists, whenever the topological mating does. See [35, 10] for various notions of conformal mating.

- Suppose P and Q are hyperbolic quadratic polynomials, and denote the corresponding centers by p_0 and q_0, let $f_0 \cong P_0 \coprod Q_0$. Now P_0 is quasiconformally conjugate to P in a neighborhood of the Julia set $\mathcal{J}_{p_0} = \partial\mathcal{K}_{p_0}$, analogously for Q_0, and there is a rational map f with the corresponding multipliers, such that f_0 is quasiconformally conjugate to f in a neighborhood of $\mathcal{J}_{f_0}$. The conjugations of polynomials respect the landing of dynamic rays, so the semi-conjugations from P_0 and Q_0 to f_0 define new semi-conjugations from P and Q to f in neighborhoods of the Julia sets. Using conformal conjugations to Blaschke products on the immediate basins, the required semi-conjugations from $\mathcal{K}_p \sqcup \mathcal{K}_q \to \widehat{\mathbb{C}}$ are constructed, and $f \cong P \coprod Q$ is a geometric mating. The same argument works when one polynomial is hyperbolic and the other one is preperiodic.

- A geometrically finite quadratic polynomial is preperiodic, hyperbolic, or parabolic. Haïssinsky–Tan have constructed all matings of geometrically finite polynomials from non-conjugate limbs [16]: when parabolic parameters are approximated radially from within hyperbolic components, the geometric matings converge. The proof is based on distortion control techniques by Cui. On the other hand, when two parabolic parameters are approximated tangentially, mating may be discontinuous; see [15, 6] and Section **??**.

- For quadratic polynomials having a fixed Siegel disk of bounded type, Yampolsky–Zakeri [48] construct the geometric mating when the multipliers are not conjugate, and obtain the mating of one Siegel polynomial with the Chebychev polynomial in addition. The proof combines Blaschke product models, complex a priori bounds, and puzzles with bubble rays.

- Suppose θ defines a parameter p with a Siegel disk of bounded type and consider the real parameter q with angle $\Theta = 1/2 + \theta/4$ according to the Douady magic formula, which is strongly recurrent. The geometric mating $f \cong P \coprod Q$ exists according to Blé-Valdez [2].

- Denote by V_2 the family of quadratic rational maps $f_a(z) = (z^2 + a)/(z^2 - 1)$ with a superattracting 2-cycle. The parameter space looks like a mating between the Mandelbrot set $\mathcal{M}$ and the Basilica Julia set $\mathcal{K}_B$, both truncated between the rays with angles $\pm 1/3$. Capture components correspond to Fatou components of the Basilica. Large classes of maps in V_2 are known to be matings of quadratic polynomials with the Basilica, by work of Luo, Aspenberg–Yampolsky, Dudko, and Yang [26, 1, 13, 49]. The basic idea is to construct puzzle-pieces with bubble rays both in the dynamic plane and in the parameter plane. This approach does not seem to generalize to V_3, because Rabbit matings may be represented by Airplane matings as well.

- When p is periodic and $\overline{q}$ shares an angle with a boundary point of a preperiodic Fatou component, the geometric mating is constructed by regluing a capture according to Mashanova–Timorin [27].

- For large classes of geometrically finite and infinite examples, Theorem 3.3 shows that ray-equivalence classes are uniformly bounded trees. So the topological mating exists according to Epstein [35], but the geometric mating is not constructed here.

In higher degrees, a topological mating $P \coprod Q$ may exist when there is no geometric mating. An example with periodic cubic polynomials is discussed in [44, 9]. Other examples are obtained from expanding Lattès maps: choose a 2×2 integer matrix A with trace t and determinant d satisfying $0 < t-1 < d < t^2/4$, e.g., $t = d = 5$. This defines a Thurston map g of type (2, 2, 2, 2) with degree d. Now g^n is expanding and not equivalent to a rational map, since the eigenvalues of A^n are real > 1 and distinct [12, 19, 22]. But according to [29], g^n is a topological mating for large n.

2.3 Ray connections and ray-equivalence classes

For the mating of quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ with locally connected Julia sets, rays and ray-equivalence classes are defined in terms of the formal mating $g = P \sqcup Q$. A ray connection is an arc within a ray-equivalence class. The length of an arc or loop is the number of rays involved, and the diameter of a ray-equivalence class is the greatest distance with respect to this notion of length. We shall discuss the structure of ray-equivalence classes in detail for various examples, and show existence of the topological mating in certain cases. By the Moore Theorem [35, 33], all ray-equivalence classes must be trees and the ray-equivalence relation must be closed. For this the length of ray connections will be more important than the number of rays and landing points in a ray-equivalence class: there is no problem when, e.g., branch points with an increasing number of branches converge to an endpoint, since the angles will have the same limit. The following results are proved in Propositions 4.3 and 4.12 of [35]:

Proposition 2.4 (Ray connections and matability, Epstein)
Consider ray-equivalence classes for the formal mating $g = P \sqcup Q$ of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, with $\mathcal{K}_p$ and $\mathcal{K}_q$ locally connected.

1. *If all classes are trees and uniformly bounded in diameter, the topological mating $P \coprod Q$ exists as a branched cover of the sphere.*

2. *If there is an infinite or a cyclic ray connection, the topological mating does not exist.*

Note that there is no statement about non-uniformly bounded trees. For preperiodic matings having a pseudo-equator, Meyer [29] has shown that ray-equivalence classes are bounded uniformly in size; hence the diameters are bounded uniformly as well. Theorem 3.3 gives topological matings $P \coprod Q$, where all ray-equivalence classes are bounded uniformly in diameter, but they need not be bounded in size; see Example 3.4.

3 Short ray connections

We shall obtain explicit bounds on ray connections in Section 3.2, discuss special irrational ray connections in Section 3.3, search long ray connections algorithmically in

Section 5, and give examples of cyclic ray connections in Section 4. The results provide partial answers to Questions 3.1–3.3, 3.5–3.7, and 3.9 in [6].

3.1 Shape of ray-equivalence classes

The following description of ray-equivalence classes can be given in general, speaking of connections between $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_{\overline{q}}$ according to Figure **??**:

Proposition 3.1 (Shape of ray-equivalence classes, following Sharland)
Consider rational and irrational ray-equivalence classes for the formal mating $g = P \sqcup Q$ of quadratic polynomials, with $\mathcal{K}_p$ and $\mathcal{K}_q$ locally connected.

1. *Any branch point of a ray-equivalence class is a branch point of $\mathcal{K}_p$ or $\mathcal{K}_{\overline{q}}$. Thus it is precritical, critical, preperiodic, or periodic. So with countably many exceptions, all ray-equivalence classes are simple arcs (finite or infinite), or simple loops.*

2. *Suppose the periodic ray-equivalence class C is a finite tree, then all the angles involved are rational of the same ray period m. Either C is an arc and m-periodic as a set, or it contains a unique point z of period $m' = m/r$ with $r \geq 2$ branches. Then z is the only possible branch point of C, so C is a topological star when $r \geq 3$.*

3. *Suppose that the topological mating $P \coprod Q$ exists. Then only critical and precritical ray-equivalence classes may have more than one branch point. More precisely, we have the following cases:*
a) Both P and Q are geometrically finite. Then irrational ray-equivalence classes of g are finite arcs, and rational ray-equivalence classes may have at most seven branch points.
b) Precisely one of the two polynomials is geometrically finite. Then irrational classes have at most one branch point, and rational classes may have up to three.
c) Both polynomials are geometrically infinite. Then irrational classes have at most three branch points, and rational classes have at most one.

Item 2 was used by Sharland [41, 42] to describe hyperbolic matings with cluster cycles. It is employed in Sections 4.3 and 6 of [22] to classify matings with orbifold of essential type (2, 2, 2, 2), and in [21] as well.

Proof: 1. Since the rays themselves are not branched, the statement is immediate from the No-wandering-triangles Theorem [47, 37] for branch points of quadratic Julia sets.

2. Rational rays landing together have the same preperiod and ray period, and only rational rays land at periodic and preperiodic points of a locally connected Julia set. So they never land together with irrational rays. Ray-equivalence classes are mapped homeomorphically or as a branched cover. If a finite tree C satisfies $g^{m'}(C) \cap C \neq \emptyset$ with minimal $m' \geq 1$, we have $g^{m'}(C) = C$ in fact, and C does not contain a critical point. Since $g^{m'}$ is permuting the points and rays of C, there is a minimal $m \geq m'$, such that g^m is fixing all points and rays, and all angles are m-periodic. Suppose first that C contains a branch point z with $r \geq 3$ branches. It is of satellite type, so its period is $m/r \geq m'$, and the r branches are permuted transitively by $g^{m/r}$. Thus all the other points are m-periodic, and they cannot be branch points, because the first return map would not permute their branches transitively. So $m' = m/r$. On the other hand, if C is an arc, then $g^{m'}$ is either orientation-preserving and $m = m'$, or orientation-reversing and $m = 2m'$. In the

latter case, the number of rays must be even, since each point is mapped to a point in the same Julia set, and the point in the middle has period $m' = m/2$.

3) A periodic ray-equivalence class may contain a single branch point according to item 2. In case a) a preperiodic class may contain two postcritical points (from different polynomials), and we have a pullback from critical value to critical point twice. Each time the number of branch points may be doubled, and a new branch point be created. This can happen only once in case b) and not at all in case c). On the other hand, an irrational ray-equivalence class C may contain only critical and precritical branch points, and this can happen only when the corresponding polynomial is geometrically infinite. Some image of C contains postcritical points instead of (pre-)critical ones, and it can contain only one postcritical point from each polynomial, since it would be periodic otherwise. So pulling it back to C again gives at most three branch points. Note that an irrational periodic class would be infinite or a loop, contradicting the assumption of matability. ∎

3.2 Bounding rational and irrational ray connections

When p is postcritically finite, every biaccessible point $z \in \partial\mathcal{K}_p$ will be iterated to an arc $[-\beta_p\,,\,\beta_P]$, then to $[\alpha_p\,,\,-\alpha_p]$, then to $[\alpha_p\,,\,p]$, and it stays within the Hubbard tree $T_p \subset \mathcal{K}_p$. In [33], Milnor discusses several aspects of the geometric and the topological mating $P \coprod Q$ with $p = q = \gamma_M(1/4)$. Every non-trivial ray connection will be iterated to a connection between points on the Hubbard trees, since every biaccessible point is iterated to the Hubbard tree T_p or $T_{\overline{q}}$. The two sides of the arcs of T_p are mapped in a certain way, described by a Markov graph with six vertices, such that only specific sequences of binary digits are possible for external angles of T_p. It turns out the only common angles of T_p and $T_{\overline{q}}$ are the 4-cycle of 3/15 and some of its preimages. This fact implies that all ray connections between the Julia sets $\mathcal{K}_p$ and $\mathcal{K}_{\overline{q}}$ are arcs or trees of diameter at most 3, so the topological mating exists by Proposition 2.4.

We shall consider an alternative argument, which is due to [44] in a cubic situation. It gives weaker results in the example of $1/4 \sqcup 1/4$, but it is probably easier to apply to other cases: T_q is obtained by cutting away the open sector between the rays with angles 9/14 and 11/14, and its countable family of preimages, from $\mathcal{K}_q$. So no $z \in T_{\overline{q}}$ has an external angle in the open interval (3/14, 5/14), or in its preimages (3/28, 5/28) and (17/28, 19/28). Now for every z on the arc $[\alpha_p\,,\,-\alpha_p]$, the angles on one side are forbidden. That shall mean that the corresponding rays do not connect z to a point in $T_{\overline{q}}$, but to an endpoint of $\mathcal{K}_{\overline{q}}$ or to a biaccessible point in a preimage of $T_{\overline{q}}$. This fact implies that every ray-equivalence class has diameter at most four, which is weaker than Milnor's result, but sufficient for the topological mating.

This argument shall be applied to another example, the mating of the Kokopelli P and the Airplane Q. Here $T_{\overline{q}} = T_q$ has no external angle in (6/7, 1/7), and one side of $[\alpha_p\,,\,-\alpha_p]$ has external angles in [1/14, 1/7]. Treating preimages of α_p separately, it follows that no other point in $\mathcal{K}_p$ is connected to two points in $T_{\overline{q}}$, and we shall see that all ray-equivalence classes are uniformly bounded trees. So the existence of the topological mating is obtained without employing the techniques of Theorem 2.2 by Thurston, Rees–Shishikura–Tan, and Rees–Shishikura. Moreover, this approach works for geometrically finite and infinite polynomials as well. E.g., q

may be any real parameter before the Airplane root, and p be any parameter in the small Kokopelli Mandelbrot set. Note however, that only the topological mating is obtained here, not the geometric mating: the method does not show that there is a corresponding rational map.

To formulate the argument when $\mathcal{K}_q$ is locally connected but Q is not postcritically finite, we shall employ a generalized Hubbard tree T_q: it is a compact, connected, full subset of $\mathcal{K}_q$, which is invariant under Q and contains an arc $[\alpha_q, q]$. If $\mathcal{K}_q$ has empty interior and q is not an endpoint with irrational angle, there will be a minimal tree with these properties. When $\mathcal{K}_q$ has non-empty interior, a forward-invariant topological tree need not exist, but we may add closed Fatou components to suitable arcs to define T_q. And when q is an irrational endpoint, we shall assume that it is renormalizable, and add complete small Julia sets to $T_{\overline{q}}$. — Note that in any case, every biaccessible point in $\mathcal{K}_q$ will be absorbed by T_q, since $[\alpha_q, q] \subset T_q$.

Proposition 3.2 (Explicit bound on ray connections)
Consider ray-equivalence classes for the formal mating $g = P \sqcup Q$ of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, with $\mathcal{K}_p$ and $\mathcal{K}_q$ locally connected, and with a generalized Hubbard tree $T_q \subset \mathcal{K}_q$ as defined above. Now suppose that there is an open set of angles, such that no external angle of $T_{\overline{q}}$ is iterated to this forbidden set, and such that for an arc $[\alpha_p, -\alpha_p] \subset \mathcal{K}_p$, the external angles on one side are forbidden. Then:

1. *Any point in $\mathcal{K}_p$ has at most one ray connecting it to a point in the generalized Hubbard tree $T_{\overline{q}}$ of $\overline{Q}$.*
2. *All ray-equivalence classes have diameter bounded by eight, since each class is iterated to a tree of diameter at most four.*
3. *Moreover, there are no cyclic ray connections, so the topological mating $P \coprod Q$ exists according to Proposition 2.4.*

Proof: 1. By assumption, α_p has at least one forbidden angle, but there may be several allowed angles. Since these are permuted transitively by iteration, none of them is connected to $T_{\overline{q}}$. In particular, there is no ray connecting α_p to $\alpha_{\overline{q}}$, so p and q are not in conjugate limbs. Suppose $z \in \partial\mathcal{K}_p$ is not a preimage of α_p. If it had two rays connecting it to points in $T_{\overline{q}}$, this connection could be iterated homeomorphically until both rays are on different sides of the arc $(\alpha_p, -\alpha_p)$, contradicting the hypothesis since $T_{\overline{q}}$ is forward-invariant. (Even if z is precritical and reaches 0 with both rays on one side, the next iteration will be injective.)

2. Suppose C is any bounded connected subset of a ray-equivalence class. Iterate it forward (maybe not homeomorphically) until all of its preperiodic points have become periodic, all critical and precritical points have become postcritical, and all biaccessible points of $\mathcal{K}_{\overline{q}}$ have been mapped into $T_{\overline{q}}$. So C is a preimage of an eventual configuration C_∞, which is a subset of a ray-equivalence class of diameter at most four, since it contains at most one biaccessible point of $T_{\overline{q}}$. E.g., it might be a periodic branch point of $\mathcal{K}_p$ connected to several endpoints of $\mathcal{K}_{\overline{q}}$, or a point of $T_{\overline{q}}$ connected to two or more biaccessible points of $\mathcal{K}_p$, which are connected to endpoints of $\mathcal{K}_{\overline{q}}$ on the other side. In general, taking preimages will give two disjoint sets of the same diameter in each step, unless there is a critical value involved.

Now C_∞ contains at most one postcritical point of $\mathcal{K}_{\overline{q}}$. If there are several postcritical points of $\mathcal{K}_p$, then C_∞ is periodic, and preperiodic preimages contain at most one postcritical point of P. So when pulling back C_∞, the diameter is increased at most twice, and it becomes at most 16. Actually, when C_∞ has diameter 4, neither postcritical point can be an endpoint of C_∞, and some sketch shows that the diameter will become at most 8.

3. If C is a cyclic ray connection, it will be iterated to a subset of a tree C_∞ according to item 2. This means that in the same step, both critical points are connected in a loop C', and $C'' = g(C)$ is a simple arc connecting the critical values $p \in \mathcal{K}_p$ and $\overline{q} \in \mathcal{K}_{\overline{q}}$. This cannot be a single ray, since p and q are not in conjugate limbs. Suppose that C'' is of the form $p - \overline{q}' - p' - \overline{q}$ with $\overline{q}' \notin T_{\overline{q}}$. Now $\overline{q}'$ is biaccessible, so it will be iterated to $T_{\overline{q}}$, and then it must coincide with an iterate of $\overline{q}$ by item 1. So C'' is not iterated homeomorphically, and p' must be critical or precritical. But then C'' would be contained in a finite periodic ray-equivalence class, and the critical value of P would be periodic, contradicting $p \in \partial\mathcal{K}_p$. The same arguments work to exclude longer ray connections between the critical values p and $\overline{q}$. ■

The following theorem provides large classes of examples. The parameter p is described by a kind of sector, and q is located on some dyadic or non-dyadic vein. More generally, q may belong to a primitive or satellite small Mandelbrot set, whose spine belongs to that vein. Let us say that q is centered on the vein:

Theorem 3.3 (Examples of matings with bounded ray connections)
When p and q are chosen as follows, with locally connected Julia sets, the topological mating $P \coprod Q$ exists according to Proposition 3.2:

a) The parameter q is in the Airplane component or centered on the real axis before the Airplane component, and p in the limb M_t with rotation number $0 < t \le 1/3$ or $2/3 \le t < 1$.

b) q is centered on the non-dyadic vein to $\mathrm{i} = \gamma_M(1/6)$*, and $p \in \mathcal{M}_t$ with rotation number $0 < t < 1/2$ or $2/3 < t < 1$.*

c) q is centered on the dyadic vein to $\gamma_M(1/4)$, and p is located between the non-dyadic veins to $\gamma_M(3/14)$ and $\gamma_M(5/14)$. This means $p \in \mathcal{M}_t$ with $1/3 < t < 1/2$, or $p \in \mathcal{M}_{1/3}$ on the vein to $\gamma_M(3/14)$ or to the left of it, or $p \in \mathcal{M}_{1/2}$ on the vein to $\gamma_M(5/14)$ or to the right of it. In particular, p may be on the vein to $\gamma_M(1/4)$, too.

Proof: The case of q in the main cardioid is neglected, because all ray connections are trivial. We shall consider the angles of $\mathcal{K}_{\overline{q}}$ according to Figure **??**. When Q has a topologically finite Hubbard tree T_q, maximal forbidden intervals of angles are found by noting that orbits entering T_q must pass through $-T_q$. See, e.g., Section 3.4 in [20]. Denote the characteristic angles of the limb $\mathcal{M}_t$ by $0 < \theta_- < \theta_+ < 1$. For $p \in \mathcal{M}_t$, the arc $[\alpha_p, \beta_p]$ has angles θ with $0 \le \theta \le \theta_+/2$ on the upper side and with $(\theta_- + 1)/2 \le \theta \le 1$ on the lower side.

a) If q is in the Airplane component or before it, the Hubbard tree is the real interval $T_q = [q, q^2 + q]$. If q belongs to a small Mandelbrot set centered before the Airplane, T_q may contain all small Julia sets meeting an arc from q to $f_q(q)$ within $\mathcal{K}_q$. Now no $z \in T_{\overline{q}}$ has an angle in $(6/7, 1/7)$. So Theorem 3.3 applies when $\theta_+/2 < 1/7$ or $(\theta_- + 1)/2 > 6/7$. The strict inequality is not satisfied for $t = 1/3$ and $t = 2/3$. Then α_p and its preimages may be connected to three points in the Hubbard tree of the Airplane, but the diameter is bounded by four as well. Note that behind case a), with $q = \gamma_M(28/63)$ and $p = \gamma_M(13/63)$, there is a ray connection of length six.

b) When q is centered on the vein to $\gamma_M(1/6)$, the interval $(11/14,\, 1/14)$ is forbidden, so $(13/14,\, 3/14)$ is forbidden for $T_{\overline{q}}$. We need $\theta_+/2 < 3/14$ or $(\theta_-+1)/2 > 13/14$.

c) For parameters q centered on the vein to $\gamma_M(1/4)$, the interval $(9/14,\, 11/14)$ is forbidden, so $(3/14,\, 5/14)$ is forbidden for $T_{\overline{q}}$. We shall take its preimage $(3/28,\, 5/28) \cup (17/28,\, 19/28)$ instead. When p is between the veins to $\gamma_M(3/14)$ and $\gamma_M(5/14)$, these two intervals are overlapping in a sense: every $z \in (\alpha_p\,,\, -\alpha_p)$ has all angles on one side in a forbidden interval. But then we have $p \in \mathcal{M}_t$ with $\theta_+/2 < 5/28$ or $(\theta_- + 1)/2 > 17/28$, so the forbidden intervals extend to $\pm\alpha_p$. ■

Example 3.4 (Bounded unlimited ray-equivalence classes)
Suppose q is chosen according to item a) or b), and p is constructed as follows. Take a primitive maximal component in the $1/3$-limb, then a primitive maximal component in its $1/4$-sublimb, a primitive maximal component in its $1/5$-sublimb ..., then the limit p has an infinite angled internal address with unbounded denominators. $\mathcal{K}_p$ is locally connected by the Yoccoz Theorem [18, 31], the topological mating exists according to Theorem 3.3, and there are branch points with any number of branches. So ray-equivalence classes are bounded uniformly in diameter, but not in size in the sense of cardinality.

3.3 More on irrational ray connections

If two parameter rays with angles $\theta_- < \theta_+$ accumulate at the same fiber of $\mathcal{M}$, it will intersect some dyadic vein in one point c, which is called combinatorially biaccessible. $\mathcal{K}_c$ is locally connected and the dynamic rays with angles $\theta_\pm$ land at the critical value c, unless c is parabolic. See the references in Section 4.4 of [20]. The following proposition shows that cyclic ray connections for matings of biaccessible parameters can exist only in special situations, since they cannot be preserved for postcritically finite parameters behind them, where they are ruled out by Theorem 2.2 of Rees–Shishikura–Tan. Compared to Proposition 3.2, the situation is more general and the conclusion is weaker.

Proposition 3.5 (Cyclic irrational ray connections)
Consider the formal mating g of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$, with parameters p and q not in conjugate limbs of $\mathcal{M}$.

a) If p is geometrically finite and q is combinatorially biaccessible, or vice versa, or both are geometrically finite, then g does not have a cyclic ray connection.

b) If both p and q are combinatorially biaccessible and not geometrically finite, then g has a cyclic ray connection, if and only if there is a ray connection between the critical values p and $\overline{q}$.

Proof: If both parameters are postcritically finite, the topological mating exists according to Theorem 2.2, and there can be no cyclic ray connection by the Moore Theorem. For hyperbolic or parabolic parameters, the ray connections will be the same as for the corresponding centers. In general, a ray connection between the critical values will have a cyclic preimage, so this connection does not exist in case a). Conversely, a cyclic connection C that does not contain precritical points of the same generation, will give a contradiction for postcritically finite parameters behind the current ones: it may be iterated, possibly non-homeomorphically, to a cyclic connection C_∞ between points on the Hubbard trees, which are not critical or precritical, and this connection C_∞ would survive. To see this for P, denote the external angles of the critical value p by $\theta_- < \theta_+$. Then no ray of C_∞ will have an angle in $(\theta_-/2\,,\,\theta_+/2) \cup ((\theta_- + 1)/2,\,(\theta_+ + 1)/2)$. For parameters c behind p, the critical point is located in a strip bounded by these four rays, so no precritical leaf can separate the rays biaccessing points of $\mathcal{K}_p$ in C_∞. (I have learned this technique from Tan Lei.) The same argument applies to $\overline{q}$ and parameters behind it. ∎

The following proposition is motivated by Question 3.7 in [6]. It deals with angles θ that are rich in base 2: the binary expansion contains all finite blocks, or equivalently, the orbit of θ under doubling is dense in $\mathbb{R}/\mathbb{Z}$. Angles with this property are rarely discussed for quadratic dynamics, but they form a subset of full measure in fact.

Proposition 3.6 (Rich angles and irrational ray connections)
Suppose the angle θ is rich in base 2. Set $\theta_n = 2^n\theta$ and $c_n = \gamma_M(\theta_n)$ for $n \ge 1$. Then c_n is a non-renormalizable endpoint of $\mathcal{M}$ with trivial fiber, $\mathcal{K}_{c_n}$ is a dendrite, and the critical orbit is dense in $\mathcal{K}_{c_n}$.

1. *For $n \neq m$ consider the formal mating g of P and Q, with $p = c_n$ and $q = \overline{c_m}$. Then g has a ray-equivalence class involving the angle θ, which is an arc of length four.* (*Note that n and m may be chosen such that p and q are not in conjugate limbs, but it is unknown whether the topological or geometric mating exists.*)

2. *Let $\mathcal{X}_\theta \subset \mathcal{M}$ contain all parameters c, such that θ is biaccessing $\mathcal{K}_c$. Then $\mathcal{X}_\theta$ is totally disconnected, and it contains $c = -2$ and all c_n. So it has infinitely many point components, and it is dense in $\partial\mathcal{M}$.*

Proof: Renormalizable and biaccessible parameters do not have dense critical orbits. The orbit of an angle at the main cardioid is confined to a half-circle [8]. By the Yoccoz Theorem [18, 31], $\mathcal{K}_{c_n}$ is locally connected with empty interior.

1. Assuming $n < m$, pull back the ray of angle θ_m connecting postcritical points of $\mathcal{K}_p$ and $\mathcal{K}_{\overline{q}}$. This ray connects two endpoints, so it forms a trivial ray-equivalence class. Since both points are postcritical of different generations, the diameter is doubled twice under iterated pullback (whenever there are two preimages, choose the component containing an image of θ).

2. For $c = -2$, every irrational angle is biaccessing, and for c_n, θ belongs to a critical or precritical point. By excluding all other cases, $\mathcal{X}_\theta$ can contain only these and maybe other non-renormalizable, postcritically infinite endpoints outside of the closed main cardioid, thus it has only point components. So suppose that θ is biaccessing $\mathcal{K}_c$:
For a Siegel or Cremer polynomial of period 1, at most precritical points or preimages of α_c are biaccessible [38], and the orbit of angles is not dense.
Pure satellite renormalizable parameters have only rational biaccessing angles outside of the small Julia sets.
When c is primitive renormalizable, the biaccessible points outside of the small Julia sets are iterated to a set moving holomorphically with the parameter, see Section 4.1 in [20]. It is contained in a generalized Hubbard tree T_c in the sense of Proposition 3.2.
When c is postcritically finite or biaccessible, all biaccessible points are absorbed by

a topologically finite tree T_c. So their orbits are not dense in K_c unless $T_c = \mathcal{K}_c$, which happens only for $c = -2$.
It remains to show that $\mathcal{X}_\theta$ is dense in $\partial\mathcal{M}$: from a normality argument it is known that β-type Misiurewicz points are dense. For any Misiurewicz point $a = \gamma_M(\widetilde{\theta})$ there is a subsequence with $\theta_n' \to \widetilde{\theta}$. Then $c_n' \to a$, since Misiurewicz points have trivial fibers [36]. ∎

4 Cyclic ray connections

First we shall construct cyclic ray connections for the formal mating g of the Airplane $P(z) = z^2 - 1.754877666$ and the Basilica $Q(z) = z^2 - 1$. See Figure 1. All biaccessing rays of Q are iterated to the angles 1/3 and 2/3 at $\alpha_q = \alpha_{\overline{q}}$. Denote by C_0 the cyclic connection formed by the rays with angles 5/12 and 7/12. Pulling it back along the critical orbit of the Airplane gives nested cycles C_n around the critical value p, since g^3 is proper of degree 2 from the interior of C_1 to the interior of C_0. Now C_n has 2^n points of intersection with $\mathcal{K}_p$, so its length is not uniformly bounded as $n \to \infty$. Moreover, C_n connects points x_n converging to $x_\infty = \gamma_p(3/7) = \gamma_p(4/7)$ to points x_n' converging to $x_\infty' = \gamma_p(25/56) = \gamma_p(31/56)$. But these four rays are landing at endpoints of the Basilica, so the landing points $x_\infty \neq x_\infty'$ on the Airplane critical value component are not in the same ray-equivalence class. Thus the ray-equivalence relation is not closed. In fact, the limit set of C_n contains the boundary of the Fatou component, which meets uncountably many ray-equivalence classes. I am not sure what the smallest closed equivalence relation, or the corresponding largest Hausdorff space, will look like: it shall be some non-spherical quotient of the Basilica, with a countable family of simple spheres attached at unique points. This Hausdorff obstruction has been obtained independently by Bartholdi–Dudko [private communication]. — More generally, we have:

Theorem 4.1 (Unbounded cyclic ray connections)
Suppose p is primitive renormalizable of period m and $\mathcal{K}_p$ is locally connected. Then there are parameters $c_* \prec c_0 \prec p$, such that for all parameters q with $\overline{q}$ on the open arc from c_* to c_0, the formal mating $g = P \sqcup Q$ has non-uniformly bounded cyclic ray connections. Moreover, these are nested such that the ray-equivalence relation is not closed. So the topological mating $P \amalg Q$ is not defined on a Hausdorff space.

Proof: In the dynamic plane of $\mathcal{K}_p$, denote the small Julia set around the critical value p by $\mathcal{K}_p^m$. There are preperiodic pinching points with $\alpha_p \preceq x_* \prec x_0 \prec x_1 \prec \mathcal{K}_p^m \prec x_1'$, such that P^m is a 2-to-1 map from the strip between x_1 and x_1' to the wake of x_0. Restricting these sets by equipotential lines in addition, we obtain a polynomial-like map, which is a renormalization of P. If the pinching points are branch points, the bounding rays must be chosen appropriately. We assume that x_1 and x_1' are iterated to x_0 but never behind it, and x_0 is iterated to x_* but never behind it. More generally, x_* may be a periodic point. The construction of these points is well-known from primitive renormalization; see [36, 46, 25].

Since the points x_* and x_0 are characteristic in $\mathcal{K}_p$, there are corresponding Misiurewicz points c_* and c_0 in $\mathcal{M}$. (If x_* is periodic, then c_* is a root.) When the parameter $\overline{q}$ is in the wake of c_*, or in the appropriate subwake, then x_0 will be moving holomorphically with the parameter and keep its external angles. When

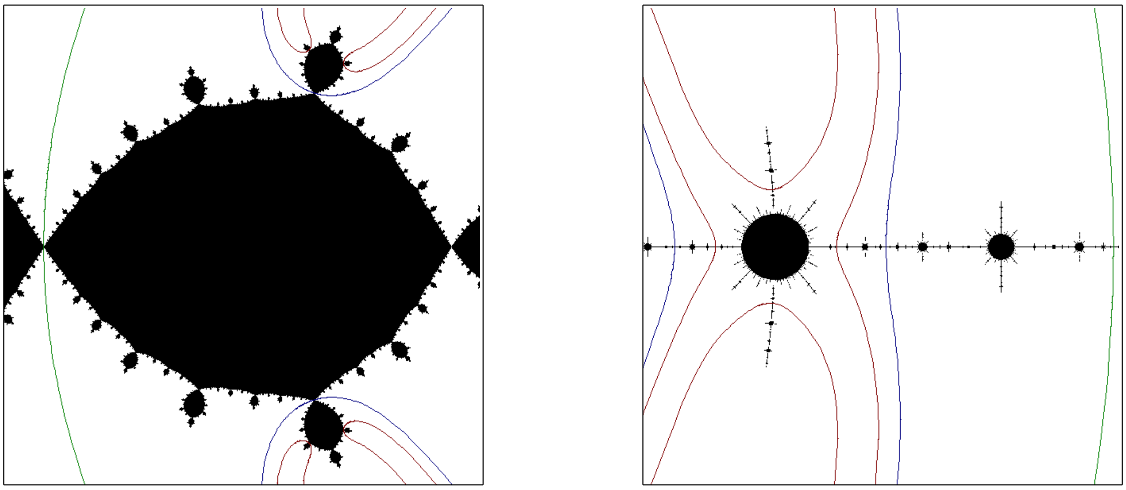

Figure 1: The formal mating g of the Airplane $\mathcal{K}_p$ (on the right) and the Basilica $\mathcal{K}_q$ (shown rotated on the left). The green ray connection C_0 has the angles $5/12$ and $7/12$. Suitable preimages C_1 (blue), C_2 (red), …form nested cycles around the critical value component of the Airplane. The nested domains are typical of primitive renormalization. The canonical obstruction of g is discussed in Figure 2 of [24].

$\overline{q}$ is chosen on the regulated arc from c_* to c_0, then $\mathcal{K}_{\overline{q}}$ will be locally connected. In $\mathcal{K}_{\overline{q}}$ the point corresponding to x_0 has the same external angles as in $\mathcal{K}_p$, and no postcritical point is at this point or behind it. Thus the four rays defining the strip between x_1, $x_1' \in \mathcal{K}_p$ are landing in a different pattern at $\mathcal{K}_{\overline{q}}$.

Now consider the formal mating $g = P \sqcup Q$. We shall keep the notation x_i, p, $\mathcal{K}_p$, $\overline{q}$, $\mathcal{K}_{\overline{q}}$ for the corresponding points and sets on the sphere. Since the two rays bounding the wake of x_0, or the relevant subwake, are landing together at $\mathcal{K}_{\overline{q}}$, they form a closed ray connection C_0. Its preimage is a single curve consisting of four rays, two pinching points in $\mathcal{K}_p$, and two pinching points in $\mathcal{K}_{\overline{q}}$. This can be seen on the sphere, since C_0 is separating the critical values of g, or in the dynamic plane of $\overline{q}$, since $\overline{q}$ is not behind the point corresponding to x_0. Now the new curve is pulled back with g^{m-1} to obtain C_1, which is a closed curve connecting x_1 and x_1' to two pinching points in $\mathcal{K}_{\overline{q}}$. By construction, g^m is proper of degree 2 from the interior of C_1 to the interior of C_0, and the former is compactly contained in the latter. g^m behaves as a quadratic-like map around $\mathcal{K}_p^m$, but only points below the equator will converge to the small Julia set under iterated pullback.

Define the curves C_n inductively; they form strictly nested closed curves and the number of rays is doubled in each step. E.g., C_2 is intersecting $\mathcal{K}_p$ in four points. The two preimages x_2 and x_2' of x_1 are located between x_1 and x_1', while the two preimages of x_1' belong to decorations of $\mathcal{K}_p^m$ attached at the points with renormalized angles $1/4$ and $3/4$. We have $x_0 \prec x_1 \prec x_2 \prec \ldots \prec \mathcal{K}_p \prec \ldots \prec x_2' \prec x_1'$. The limits x_∞ and x_∞' are the small β-fixed point of $\mathcal{K}_p^m$ and its preimage, the small $-\beta$. Now x_n and x_n' are connected by C_n, but x_∞ and x_∞' are not ray-equivalent, because the former is periodic and the latter is preperiodic. ∎

More generally, $\overline{q}$ may be any parameter in the strip between c_* and c_0, as long as its critical orbit does not meet the point corresponding to x_0 or get behind it. — Note that by taking iterated preimages of a finite ray-equivalence tree, you will merely get uniformly bounded trees: the diameter can be increased only when a critical value is pulled back to a critical point, which can happen at most twice according to Proposition 3.1: a finite irrational tree cannot be periodic, so it does not contain more than one postcritical point from each polynomial.

5 Long ray connections

Consider rational ray-equivalence classes for the formal mating $g = P \sqcup Q$ with parameters p, q in non-conjugate limbs of $\mathcal{M}$. A non-trivial periodic ray connection requires pinching points in $\mathcal{K}_p$ and $\mathcal{K}_q$ with specific angles, which exist if and only if the parameters p, q are at or behind certain primitive roots or satellite roots. So a longer ray connection means that there are several relevant roots before the current parameters, and on the same long vein in particular. Let us say that a ray connection is **maximal**, if it is not part of a longer connection existing for parameters behind the current ones. The following ideas were used to determine all maximal ray connections algorithmically for ray periods up to 24; see Table 1.

Per.	length 5	length 6	length 7	length 8	length 10	length 12
10	32 + 0	14 + 88	—	0 + 2	—	—
11	76 + 0	20 + 0	—	—	—	—
12	46 + 0	24 + 264	—	—	—	—
13	226 + 0	72 + 0	2 + 0	2 + 0	—	—
14	285 + 0	102 + 484	4 + 0	0 + 14	0 + 2	—
15	540 + 0	192 + 184	—	—	—	—
16	958 + 0	338 + 1060	4 + 0	2 + 10	0 + 4	—
17	1872 + 0	584 + 0	14 + 0	2 + 0	—	—
18	2814 + 0	884 + 2672	22 + 0	6 + 24	0 + 8	—
19	5856 + 0	1650 + 0	26 + 0	6 + 0	—	—
20	9534 + 0	2890 + 5244	58 + 0	4 + 42	0 + 8	—
21	16978 + 0	4900 + 898	64 + 0	4 + 0	—	—
22	30180 + 0	8423 + 10928	126 + 0	18 + 132	0 + 20	0 + 2
23	55676 + 0	15300 + 0	172 + 0	18 + 0	—	—
24	95830 + 0	25968 + 25312	242 + 0	24 + 96	0 + 28	—

Table 1: The length of maximal periodic ray connections depending on the ray period. The first number counts unordered pairs of periodic parameters with primitive-only connections, the second number is the connections including a satellite cycle. Length ≤ 4 is ubiquitous, length 5 appears already for periods 7 and 9, while length 6 happens for periods 4 and 6–9 as well. Length 9 and 11 was not found for periods ≤ 24.

- Suppose $\mathcal{R}(\theta_1)$–z_p–$\mathcal{R}(\theta_2)$–z_q–$\mathcal{R}(\theta_3)$ is a step in the ray connection, then θ_1 and θ_2 belong to a cycle of angle pairs for $\mathcal{K}_p$, so there is a root before p with characteristic angles iterated to θ_1 and θ_2. Likewise, there is a root before $\overline{q}$, whose characteristic angles are iterated to θ_2 and θ_3. Conversely, given the angles $\theta_\pm$ of a root before p, we may determine conjugate angles for iterates of θ_+ under doubling, and check whether the root given by an angle pair is before $\overline{q}$; it is discarded otherwise. So we record only the angle pairs of roots, and forget about the number of iterations and about which class in a cycle contains which characteristic point. Note that there is an effective algorithm to determine conjugate angles [5, 25], probably due to Thurston.

- A maximal ray connection should be labeled by highest relevant roots on the respective veins. However, a brute-force search starting with these roots will be impractical: varying both p and $\overline{q}$ independently is too slow, and searching $\overline{q}$ depending on p requires to match different combinatorics on two sides, since the characteristic point z_p corresponding to the highest root may be anywhere in the ray-equivalence class. So the idea is to run over all roots p_1, try to build a maximal ray connection on one side of the corresponding characteristic point, and to quit if the connection can be continued on the other side of that point.

- When a pinching point of satellite type is reached under the recursive application of the conjugate angle algorithm, we may double the length and stop. Alternatively, two separate algorithms may be used, one finding primitive-only ray connections starting from the first pinching point, and another one starting with the satellite-type point in the middle of the periodic ray-equivalence class.

For period 22, this algorithm has recovered the example given to Adam Epstein by Stuart Price [6]: for p behind $\{1955623/4194303,\ 1955624/4194303\}$ and q behind $\{882259/4194303,\ 882276/4194303\}$ there is a periodic ray-equivalence class of diameter 12. For 1/2-satellites only, the same algorithm was used for periods up to 40 in addition; this produced another example of diameter 14 for period 32, with p behind $\{918089177/4294967295,\ 918089186/4294967295\}$ and q behind $\{1998920775/4294967295,\ 1998920776/4294967295\}$. Note that, e.g., taking p and q as the corresponding centers, the formal mating will have non-postcritical long ray connections and the geometric mating shows clustering of Fatou components. For suitable preperiodic parameters behind these roots, the formal mating has long periodic ray-equivalence classes with postcritical points from both orbits, and preperiodic classes may have twice or up to four times the diameter of the periodic classes. — There are several open questions on long ray connections:

- What are possible relations between the linear order of roots on the veins to p and q, and the order of pinching points within a ray-equivalence class?

- For the lowest period with a particular diameter of a ray-equivalence class, is there always a 1/2-satellite involved?

- Is there a whole sequence with similar combinatorics and increasing diameters? If it converges, does the limit show non-uniformly bounded ray connections? Does the geometric mating of the limits exist? If not, does it have infinite irrational ray connections?

- Are there only short ray connections for self-matings and for matings between dyadic veins of small denominator, even though the Hausdorff dimension of biaccessing angles is relatively high according to [14]?

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The program Mandel provides several interactive features related to the Thurston Algorithm. It is available from www.mndynamics.com . A console-based implementation of slow mating is distributed with the preprint of [24].