Ray connections and shared matings

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Abstract

There is an alternative construction of mating, when at least one polynomial is preperiodic: shift the infinite critical value of the other polynomial to a preperiodic point. Taking homotopic rays, it gives simple examples of shared matings. Sequences with unbounded multiplicity of sharing, and slowly growing preperiod and period, are obtained both in the Chebyshev family and for Airplane matings. Using preperiodic polynomials with identifications between the two critical orbits, an example of mating discontinuity is described as well.

1 Introduction

In general both $\mathcal{K}_p$ and $\mathcal{K}_q$ contain pinching points and branch points with several rays landing together, so there are ray-equivalence classes consisting of subsequent rays connecting points in $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_q$ alternately. For rational angles, the landing pattern is understood combinatorially, and the identifications of periodic and preperiodic points can be determined. Consider the example of the 5-periodic p with the external angle $11/31$ and preperiodic q with angle $19/62$ in Figure 1: since q belongs to the $2/5$-limb of $\mathcal{M}$, there are five branches of $\mathcal{K}_q$ and five external rays at the fixed point α_q, which are permuted with rotation number $2/5$ by Q. Now p is chosen such that the complex conjugate angles land pairwise with another 5-cycle at the Fatou basins; the rays of Q corresponding to the latter angles land at endpoints, including the iterate $Q(q)$ of the critical value. So in the topological mating and in the geometric mating $f \cong P \coprod Q$, the point $Q(q)$ is identified both with α_q and with a repelling 5-cycle of P. Now the critical point 0 of f is 5-periodic, while $f^2(\infty)$ is fixed. The five components of the immediate attracting basin all touch at this fixed point with rotation number $2/5$, although they had disjoint closures in $\mathcal{K}_p$.

Simple examples of shared matings and of mating discontinuity are obtained in **Section ??**. The rational map f above belongs to the same one-parameter family as matings with the Chebyshev polynomial, but it is not of this form. There are five other representations as a mating: take the Rabbit with rotation number $2/5$ for P and suitable preperiodic parameters $q_1, \dots, q_5$ for Q, which are related to the angles at $-\alpha_p$. More generally, we have $P \coprod Q_i = P \coprod Q_j$ for all p in the small satellite Mandelbrot set, since the rays at $-\alpha_p$ are homotopic with respect to the postcritical set and so the encaptures are combinatorially equivalent. Taking higher rotation numbers gives shared matings with larger multiplicity. While it is obvious that a hyperbolic rational map has only a finite number of representations as a

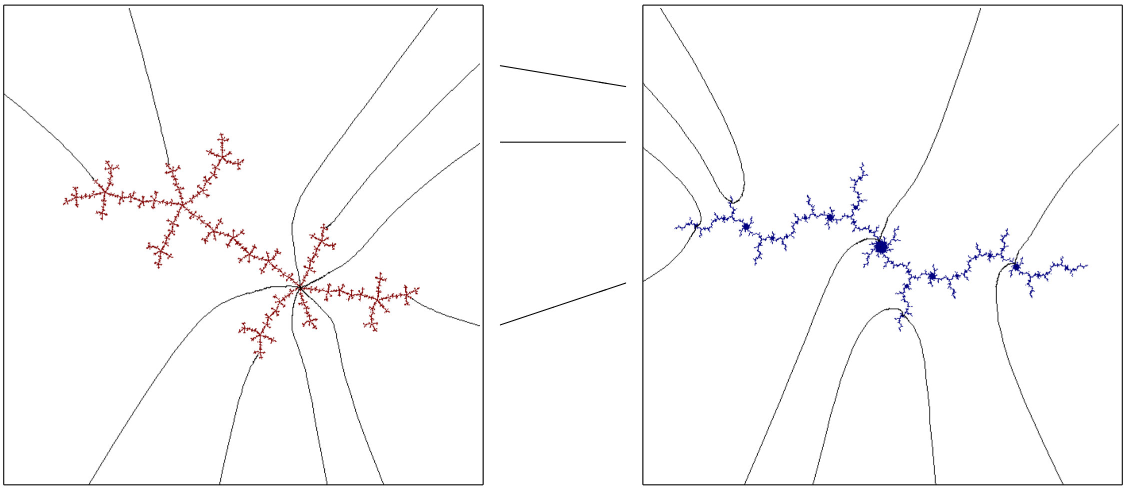

Figure 1: A formal mating $g = P \sqcup Q$. The Julia set $\mathcal{K}_p$ for the five-periodic center p corresponding to $\gamma_M(11/31)$ is shown on the right; the Misiurewicz Julia set $\mathcal{K}_q$ with $q = \gamma_M(19/62)$ in the left image is rotated. (This does not change the set itself, but its external rays.) The ray connections and dynamics are discussed in the main text.

There are various ways to visualize the sets $\varphi_0(\mathcal{K}_p)$, $\varphi_\infty(\mathcal{K}_q) \subset \widehat{\mathbb{C}}$ in the plane $\mathbb{C}$: instead of $\mathcal{K}_q$ coming from ∞, we may rotate the sphere such that $\mathcal{K}_q$ is translated above or below $\mathcal{K}_p$, or to save space here, translated to the left or right and rotated.

In any case, $\varphi_0(\mathcal{R}_p(\theta))$ is connected with $\varphi_\infty(\mathcal{R}_q(-\theta))$; three connections are indicated between the two images. When discussing the combinatorics of a ray-equivalence class, we may avoid conjugation of several angles by assuming that $\mathcal{R}_p(\theta)$ connects to $\mathcal{R}_{\overline{q}}(\theta)$, but to draw these rays without crossing, you would need to stack two sheets of paper.

mating, this is not known in general when one or both of the critical points are preperiodic. Finiteness is shown here for Chebyshev maps with one critical point periodic, and in [16] for Lattès maps. Examples with arbitrarily high multiplicity are obtained as well for matings of the Airplane with preperiodic polynomials; here preperiod and period are of the same order as the multiplicity, in contrast to the hyperbolic examples by Rees [32], where the period grows exponentially. — Simple ray connections can be used to define preperiodic matings with $f(0) = \infty$. This property is lost when preperiodic parameters converge to a parabolic parameter, confirming that mating is not jointly continuous. The mechanism is similar to geometrically infinite examples by Blé–Valdez–Epstein [3, 11], but here all maps are geometrically finite and matability does not require special arguments.

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2 Background

2.1 Polynomials

Proposition 2.1 (Douady Magic Formula, Blé)
Suppose $\theta \in [0,\, 1/3]$ *is an external angle of the main cardioid, then* $\Theta = 1/2 + \theta/4 \in [1/2,\, 7/12]$ *is an external angle of the real axis* $\mathcal{M} \cap \mathbb{R}$*.*

Proof: According to [7], the orbit of θ under doubling is confined to $[\theta/2\,, (1+\theta)/2]$. Now taking a suitable preimage shows that the orbits of θ and Θ never enter $((\theta+1)/4,\, (\theta+2)/4) \supset (1-\Theta,\, \Theta)$, so Θ is combinatorially real: it defines a unique real parameter c by approximation, and the parameter ray $\mathcal{R}_{M}(\Theta)$ accumulates at a fiber [33] intersecting the real line in c. Blé [2] has shown that f_c is strongly recurrent but not renormalizable, so the fiber is trivial and the ray actually lands, $c = \gamma_{M}(\Theta)$. ∎

2.2 Matings

Thurston, essential equivalence

2.3 Ray connections

2.4 Encaptures and matings

Theorem 2.2 (Matings as encaptures)
Suppose P *is postcritically finite and* θ *is preperiodic, such that* $q = \gamma_M(-\theta)$ *is not in the conjugate limb and* $z_1 = \gamma_p(\theta) \in \partial\mathcal{K}_p$ *is not postcritical. Then the encapture* $g_\theta = \varphi_\theta \circ P$ *along* $\mathcal{R}_p(\theta)$ *is combinatorially equivalent or essentially equivalent to the geometric mating* f *defined by* $P \coprod Q$*.*

3 Shared matings

Mating provides a partial map from $\mathcal{M} \times \mathcal{M}$ to the moduli space of quadratic rational maps. This map is neither surjective, injective, nor continuous. The characterization of matings in terms of equators and pseudo-equators by Thurston–Wittner and Meyer is discussed in [20]. Old and new examples of shared matings are described in Section 3, and particular sequences with arbitrarily high multiplicity are obtained in Sections 4 and 5. Epstein has given various examples of mating discontinuity, which are described in Section 6, and a simple new construction is presented.

A shared mating is a geometric mating with different representations, $P_1 \coprod Q_1 \simeq f \simeq P_2 \coprod Q_2$ with $P_1 \neq P_2$ or $Q_1 \neq Q_2$. There are the following examples of shared matings, and techniques for constructing them:

- Wittner [39] introduced the notion of shared matings and discussed them for V_3 in particular. A simple example is given by the geometric mating of Airplane and Rabbit, which is linear conjugate to the geometric mating of Rabbit and Airplane, $A \coprod R \simeq R \coprod A$. (Moreover, it is conjugate to a symmetric map, which is not a self-mating.) Since the two polynomials are interchanged, this example is called the Wittner flip. It can be explained by finding two different equators, which has a few generalizations:

- Exall [12] constructs pairs of polynomials P, Q with $P \amalg R \simeq Q \amalg A$ from a second equator. Using symbolic dynamics, this can be done algorithmically.
- Rees [32] uses symbolic dynamics again to obtain unboundedly shared Airplane matings. The period grows exponentially with the multiplicity.
- Denote the rabbit of rotation number k/n by R. There are $n-2$ primitive hyperbolic polynomials Q of period n, such that $\overline{Q}$ has a characteristic angle from the cycle of α_r. Then the rational map $f \cong R \amalg Q$ has a cluster cycle: both n-cycles of Fatou components have a common boundary point, which is a fixed point corresponding to α_r. Sharland [35, 36] has shown that f is determined uniquely by the rotation number and the relative displacement of the critical orbits; f has precisely two representations as a mating, which are of the form $f \cong R \amalg Q \simeq P \amalg R$.

When f is a Lattès map, different representations are known except in the case d) of $1/6 \amalg 1/6$. The Shishikura Algorithm can be used to identify the particular map f in the case of type (2, 2, 2, 2), and we have only one quadratic map of type (2, 4, 4). Combinatorial arguments show that there are basically nine, respectively three, matings of these types; see Sections 3 and 5 in [16].

- Case a) of type (2, 2, 2, 2) is given by $1/12 \amalg 5/12 \simeq -1/12 \amalg 5/12$.
- Case b) of type (2, 2, 2, 2) is $1/4 \amalg 1/4 \simeq 23/28 \amalg 13/28 \simeq 13/28 \amalg 23/28 \simeq 53/60 \amalg 29/60 \simeq 29/60 \amalg 53/60$.
- Case c) of type (2, 2, 2, 2) is $1/6 \amalg 5/14 \simeq 5/14 \amalg 1/6 \simeq 3/14 \amalg 3/14 \simeq 3/14 \amalg 1/2 \simeq 1/2 \amalg 3/14 \simeq 5/6 \amalg 1/2 \simeq 1/2 \amalg 5/6$.
- Type (2, 4, 4) is given by $\pm 1/4 \amalg 1/2 \simeq 5/12 \amalg \pm 1/6 \simeq 13/28 \amalg \pm 3/14$.

The following technique for producing shared matings is based on the representation of matings as repelling-preperiodic captures according to Theorem 2.2.

Proposition 3.1 (Shared matings from encaptures)
Suppose $P(z) = z^2 + p$ is geometrically finite, with $p \neq -2$, $p \neq 1/4$, and p not in the main cardioid. There are countably many pairs of preperiodic angles θ_1, θ_2 such that: the corresponding dynamic rays land together at a preperiodic pinching point $z_1 \in \partial\mathcal{K}_p$, which is not postcritical and not in the same branch at α_p as p, and the branch or branches of z_1 between these rays do not contain postcritical points of P or iterates of z_1. Then we have $P \amalg Q_1 \simeq P \amalg Q_2$ with $q_i = \gamma_M(-\theta_i)$. Moreover, $P \amalg Q_1 \cong P \amalg Q_2$ if β_p is not between these rays.

Proof: We need to exclude $p = 1/4$ and the main cardioid, because $\mathcal{K}_p$ would have no pinching points, and $p = -2$, because rays landing together at the interval $\mathcal{K}_{-2}$ are never homotopic with respect to the postcritical set. If P is postcritically finite, Proposition **??** shows that the encaptures $\varphi_{\theta_1} \circ P$ and $\varphi_{\theta_2} \circ P$ are combinatorially equivalent. So the canonical obstructions and the essential maps are equivalent as well. According to the proof of Theorem 2.2, given in [18], the essential maps are equivalent to the geometric matings. By continuity according to Section **??**, the result extends to geometrically finite P: ∎

- The example $11/24 \amalg 13/56 \cong 11/24 \amalg 15/56$ enjoys the following property: the latter mating has an equator and a simple pseudo-equator, while the former does not have either.

- As another example, consider $p = \gamma_M(59/240)$ and $q = \gamma_M(63/240)$. Applying this construction to P and to Q gives $P \amalg P \cong P \amalg Q$ and $Q \amalg P \cong Q \amalg Q$. Here the first and second polynomials may be interchanged on both sides, so we have four representations of the same rational map; in particular there are shared self-matings $P \amalg P \cong Q \amalg Q$, and the flipped matings $P \amalg Q \cong Q \amalg P$.

- When P is the Basilica, all pinching points are preimages of α_p. Since none of these is iterated behind itself, shared matings are obtained from any pinching point z_1, which is not α_p or behind it. Dudko [10] has shown that these are the only shared Basilica matings, since the parameter space is described as a mating of $\mathcal{M}$ and $\mathcal{K}_p$. The simplest example is given by $P \amalg (z^2 \pm \mathrm{i})$: the geometric matings are distinct and complex conjugate, and both linear conjugate to $\frac{z^2+2}{z^2-1}$. The example $P \amalg 5/24 \simeq P \amalg 7/24$ is illustrated with a video of slow mating on www.mndynamics.com . Aspenberg [1] constructs the semi-conjugation from the Basilica to the rational map, beginning with the Boettcher map; in this alternative approach, shared matings are obtained from a non-unique labeling of Fatou components by bubble rays.

- Shared matings in the family of Chebyshev maps are discussed in Section 4. In certain cases, lower bounds on the multiplicity are obtained from homotopic rays according to Proposition 3.1, or upper bounds are obtained directly.

- When z_1 is a branch point of $\mathcal{K}_p$, there may be more than two parameters q_i. In Theorem 5.1 of Section 5, unboundedly shared Airplane matings with small preperiods and periods are constructed. Although the Airplane does not contain any branch point, this is achieved by choosing q_i with a common branch point in $\mathcal{K}_q$.

- If f is a critically preperiodic rational map of degree $d \geq 2$ with three or four postcritical points, a pseudo-equator may produce several unmatings by choosing different pseudo-isotopies to its preimage [24]. A higher multiplicity is obtained when there are degenerate critical points, or when a critical point is mapped to another one. Probably the only quadratic example is the Lattès map of type (2, 4, 4). See [13] for related results on NET maps.

Remark 3.2 (Finite multiplicity)
If f is a postcritically finite quadratic rational map, can there be infinitely many representation as a mating $f \simeq P \amalg Q$?
• *When f is hyperbolic, there are only finitely many candidates for P and Q, since there are only finitely many quadratic polynomials with a given superattracting period.*
• *When one critical point is periodic and one is preperiodic, finiteness is not obvious. For a specific family, finiteness is shown in Theorem 4.2 of the following section, using similar techniques as in the Lattès case.*
• *When both critical points are preperiodic, finiteness is shown for Lattès maps*

in [16]. Probably the techniques can be applied to a few other examples of small preperiod and period, but a general proof shall be harder.

4 Shared matings in the Chebyshev family

Let us define a Chebyshev map as a quadratic rational map of the form $f(z) = f_a(z) = \frac{-z^2+a+2}{z^2+a}$, $a \neq -1$, for which $f(\infty)$ is pre-fixed: $\infty \Rightarrow -1 \rightarrow 1 \uparrow$. This family contains matings with the Chebyshev polynomial $T(z) = z^2 - 2$ in particular:

Proposition 4.1 (Chebyshev maps as matings)
Suppose $P(z) = z^2+p$ and $Q(z) = z^2+q$ are geometrically finite and not in conjugate limbs of the Mandelbrot set $\mathcal{M}$. Then the geometric mating is linear conjugate to a Chebyshev map, $f_a \simeq P \coprod Q$, if and only if P and Q are one of the following forms:

a) Q is the Chebyshev polynomial $T(z) = z^2 - 2$ and p is not in the $1/2$-limb of $\mathcal{M}$.

b) p is in the k/r-limb of $\mathcal{M}$, and $q = \gamma_M(-\theta)$, where θ is one of the r angles at $-\alpha_p \in \mathcal{K}_p$ (which depend only on k/r).

c) For a rotation number $k/r \neq 1/2$, denote the angles of the k/r-wake by $\theta_\pm$ and let $\theta = (\theta_- + \theta_+)/2$ be the unique angle of preperiod 1 and period r in that limb. If $q = \gamma_M(\theta)$, then P must be in the closed wake of the primitive hyperbolic component Ω with the root $\gamma_M(-2\theta)$.

The Petersen transformation [25] maps symmetric rational maps to Chebyshev maps, such that self-matings are mapped to Chebyshev matings; see also Remark X in [20]. In the previous section the example of shared self-matings $59/240 \coprod 59/240 \cong 63/240 \coprod 63/240$ was obtained from Proposition 3.1; now the Petersen transformation gives the shared Chebyshev mating $59/240 \coprod 1/2 \cong 63/240 \coprod 1/2$.

Proof of Proposition 4.1: As explained in Figure 1, instead of saying that angles of $z \in \mathcal{K}_p$ and $w \in \mathcal{K}_q$ are complex conjugate, we may say that $z \in \mathcal{K}_p$ shares an angle with $\overline{w} \in \mathcal{K}_{\overline{q}}$, or we may connect $\mathcal{K}_{\overline{p}}$ to $\mathcal{K}_q$ as well. In the formal mating $g = P \sqcup Q$, the ray-equivalence class of $g^2(\infty)$, corresponding to $\overline{Q}^2(0) = \overline{Q}(\overline{q})$, is fixed. By Proposition **??**, it must contain a fixed point of P or $\overline{Q}$. If this is β_p or $\beta_{\overline{q}}$, the fixed class is the 0-ray and $\overline{Q}(\overline{q}) = \beta_{\overline{q}}$, which is case a).

b) Now suppose that $\overline{Q}(\overline{q})$ is in the same ray-equivalence class as α_p and $p \in \mathcal{M}_{k/r}$. Then the critical value $\overline{q}$ is connected to $-\alpha_p$. This connection must be direct, since $\mathcal{K}_p$ does not contain another pinching cycle of ray period r. So $\overline{q}$ shares an external angle with $-\alpha_p$ and all of these angles may occur, since none is in the same sector at α_p as the critical value p, and q is not in the conjugate limb. The r angles belong to different Misiurewicz points in fact, since otherwise some $P \sqcup Q$ would have a closed ray connection.

c) Consider $q \in \mathcal{M}_{k/r}$ and $\overline{P}$ such that $Q(q)$ is in the same ray-equivalence class as α_q. The points are not equal, because the preperiod would have to be $\geq r > 1$. So the ray connection must have length 2, since length ≥ 4 would require additional pinching cycles of ray period r in $\mathcal{M}_{k/r}$. Thus q has the external angle θ defined above, and $\mathcal{K}_{\overline{p}}$ must contain a pinching cycle of period and ray period r, which connects the cycle of $2\theta = \theta_- + \theta_+$ to that of $\theta_\pm$. This cycle of $\mathcal{K}_{\overline{p}}$ persists from a primitive hyperbolic component $\overline{\Omega}$ before $\overline{p}$.

It remains to show that $\overline{\Omega}$ exists and is unique. In the dynamic plane of Q, the r rays landing at α_q define r sectors $W_1, \dots, W_r$ with $0 \in W_r$ and $q \in W_1$, such that Q is a conformal map $W_1 \to W_2 \to \dots \to W_{r-1} \to W_r$ and the sectors are permuted with rotation number k/r. The external rays with angles $2^{i-1}\theta_\pm$ bound W_i for $1 \le i \le r$. Now W_i contains $2^{i-1}\theta$ as well for $2 \le i \le r-1$ and W_r has both $2^{r-1}\theta$ and $2^r\theta = -\theta$. For $r \ge 3$ it follows that 2θ has exact period r. We are looking for a primitive orbit portrait [27] connecting each angle in $\{2^i\theta \,|\, 1 \le i \le r\}$ to a unique angle in $\{2^i\theta_- \,|\, 1 \le i \le r\} = \{2^i\theta_+ \,|\, 1 \le i \le r\}$.

Starting in W_r, connect $2^{r-1}\theta$ to either $2^{r-1}\theta_-$ or to $2^{r-1}\theta_+$, such that $2^r\theta$ is not separated from the other angles. Pull the connection back until 2θ is connected to $2\theta_-$ or $2\theta_+$. The complement of the $r-1$ disjoint small sectors is connected, so we can connect the remaining angles $2^r\theta$ and θ_+ or θ_- as well. This construction gives a valid orbit portrait and defines $\overline{\Omega}$, which has the external angles 2θ and $2\theta_-$ or $2\theta_+$. Note that it is a narrow component, i.e., its angular width is $1/(2^r-1)$ and there is no component of period $\le r$ behind $\overline{\Omega}$. To show that $\overline{\Omega}$ is unique, suppose we had started by connecting $2^{r-1}\theta$ with an angle not bounding W_r and pulled it back. This pullback would follow the rotation number k/r as well and the small sectors would overlap, the leaves would be linked. ∎

Case b) provides maps from limbs of $\mathcal{M}$ to the Chebyshev family, which are partially shared according to Proposition 3.1: e.g., for P geometrically finite in the 1/2-limb, consider the geometric matings corresponding to $P \amalg \pm 1/6$, i.e. $p \mapsto f_a \simeq P \amalg \pm 1/6$. These two maps agree on the small Mandelbrot set of period 2, but in general do not agree on its decorations. Likewise, for p in the 1/3-limb, we have three maps corresponding to $P \amalg 3/14$, $P \amalg 5/14$, and $P \amalg 13/14$, which agree on the small Mandelbrot set of period 3. In the decorations, two of the maps may agree on certain veins, but in general the third one will be different: the relevant rays are no longer homotopic. Note that according to case c), some of these Chebyshev maps are represented by $\widetilde{P} \amalg 3/14$ as well, with $\widetilde{p}$ in the Airplane wake. In particular, we have $1/7 \amalg 3/14 \cong 1/7 \amalg 5/14 \simeq 1/7 \amalg 13/14 \simeq 3/7 \amalg 3/14$. Under the Petersen transformation mentioned above, this Chebyshev map is the image of $1/7 \amalg 3/7 \simeq 3/7 \amalg 1/7$, which is a symmetric map but not a self-mating.

Theorem 4.2 (Chebyshev maps as shared matings)
Matings $P \amalg Q$ in the Chebyshev family with hyperbolic P have non-uniformly bounded multiplicity:

1. *Suppose $f = f_a$ is a Chebyshev map, such that $z = 0$ is n-periodic. Then there are at most a finite number of representations $f_a \simeq P \amalg Q$.*

2. *For each rotation number k/r, there is a unique Chebyshev map $f = f_a$, such that $z = 0$ is r-periodic and the fixed point $1 = f^2(\infty)$ is a common boundary point of the r immediate basins, which are permuted with rotation number k/r. This map has precisely $r+1$ realizations as a geometric mating, $f \simeq P \amalg Q$, when $r \ge 3$; for $r = 2$ there are only 2 representations.*

Proof: 1. P will be n-periodic, so there are only finitely many possibilities for P. We must see that r is bounded in cases b) and c). But in both cases we have $r \le n$, since the wakes of period r are narrow: in case b) this is a basic property of limbs, and in case c) it was noted in the proof of Proposition 4.1.

2. In case a) of Proposition 4.1, $z = 1$ corresponds to the ray-equivalence class

of angle 0, which does not touch a hyperbolic component of P. In cases b) and c), the rotation number at -1 is precisely k/r, so the value of k/r from the proposition must be the same as in the hypothesis of the theorem; case c) is excluded for $k/r = 1/2$. In both cases, there is only one hyperbolic component of period r in the limb or wake. It remains to show that f_a is unique, so that the $r+1$ (or two) matings actually give the same map. Intuitively, this follows from the fact that the hyperbolic component of f_a bifurcates from the hyperbolic component where -1 is attracting; the multiplier map with $\rho = \frac{-4}{a+1}$ is injective for $|a+1| \geq 4$. It can be proved by Thurston rigidity, since there is a forward-invariant graph connecting the postcritical points, which depends only on k/r up to isomorphy. So all possible maps f_a are combinatorially equivalent, linear conjugate, and equal. Note that the case of $k/r = 2/5$ was discussed in the Introduction and illustrated in Figure 1. ■

5 Unboundedly shared Airplane matings

Denoting the Rabbit by R and the Airplane by A, we have seen in the previous Section 4 that $R \coprod 3/14 \cong R \coprod 5/14 \simeq R \coprod 13/14 \simeq A \coprod 3/14$. This example belongs both to the Chebyshev family and to the family V_3 with a 3-periodic critical point. Unboundedly shared matings were obtained in Theorem 4.2.2 by increasing both the period of the hyperbolic polynomial P and the ray period of the Misiurewicz polynomial Q. Another example is obtained below, where Q is always the Airplane, and the preperiod of P is unbounded. The proof will be a simple application of Proposition 3.1 again. Airplane matings with unbounded multiplicity are due to Rees [32] with hyperbolic polynomials P, such that the period of P grows exponentially with the multiplicity.

Theorem 5.1 (Unboundedly shared Airplane matings)
For the Airplane q and $n = 3, 5, 7, \ldots,$ there are n Misiurewicz parameters $p_*, p_2, \ldots, p_n$ such that the geometric matings agree, $f \cong P_i \coprod Q$ for all $i = *, 2, \ldots, n$. Here all p_i have preperiod $n+1$, p_* has period 1 and $p_2, \ldots, p_n$ have period n; so $f(\infty)$ has preperiod $n+1$ and period 1. The statement remains true for large n, when q is any geometrically finite parameter behind $\gamma_M(5/12)$ and before the Airplane. E.g., q may be the Misiurewicz point $\gamma_M(41/96)$ as well.

Proof: Denote the Airplane parameter by q and fix $n \in \{3, 5, 7, \ldots\}$; let c be the first center of period n behind the Misiurewicz point $\gamma_M(5/12)$. The orbit of the characteristic point z_1 is ordered as

$$\begin{aligned} &z_1 < \gamma_c(5/12) < z_{n-1} < z_{n-3} < \ldots < z_6 < z_4 < \alpha_c < \\ &< z_3 < z_5 < \ldots < z_{n-2} < z_n < 0 < -\alpha_c < z_2 \, ; \end{aligned} \tag{1}$$

the critical orbit (z_i^*) is similar with $z_n^* = 0$. This ordering is well-known from discussions of Šharkovskiĭ combinatorics. It can be checked with dynamic angles as follows: first, note that the order of the critical orbit is compatible with the assumption that $f_c : [z_1^*, 0] \to [z_1^*, z_2^*]$ is strictly decreasing and $f_c : [0, z_2^*] \to [z_1^*, z_3^*]$ is strictly increasing, so this defines a unique real polynomial. Let Θ_1 be the larger angle at z_1 and denote its iterates under doubling by Θ_i. Then

$$0 < \Theta_2 < 1/6 < \Theta_n < \Theta_{n-2} < \ldots < \Theta_5 < \Theta_3 < 1/3 < \tag{2}$$

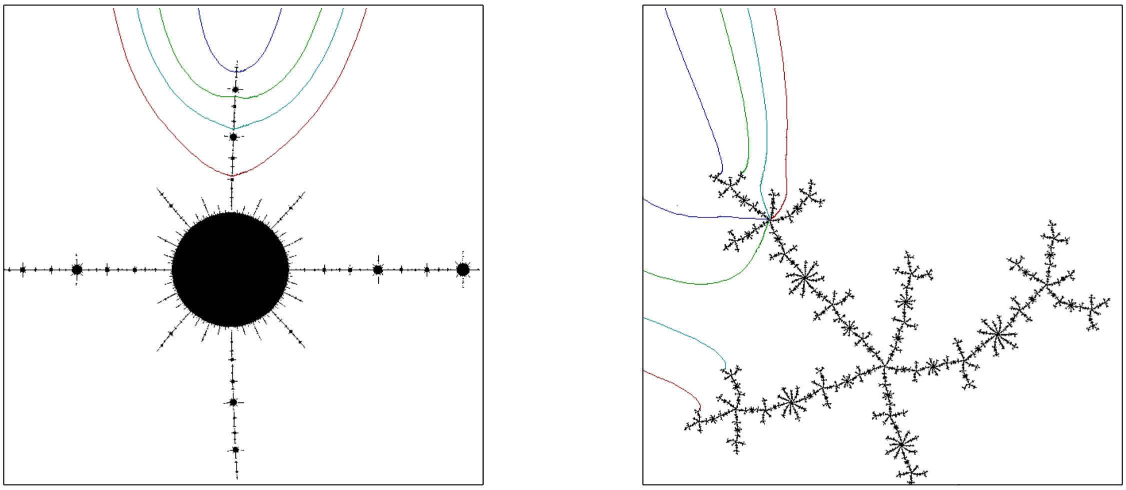

Figure 2: Consider the formal mating $g = P \sqcup Q$, with the Airplane $\mathcal{K}_q$ shown rotated on the left, and $\mathcal{K}_p$ on the right for some p in the 2/5-limb. According to the proof of Theorem 5.1, there are eight angles $\theta_2, \dots, \theta_5, \theta_2', \dots, \theta_5'$, such that $-\theta_i$ and $-\theta_i'$ land together at the Airplane $\partial\mathcal{K}_q$, while θ_i land together at $\partial\mathcal{K}_p$. So the eight rays belong to a preperiodic ray-equivalence class of diameter four; actually there are two more rays crossing the Airplane on the real axis. Now there are five parameters $p = p_*, p_2, \dots, p_5$, such that this ray-equivalence class contains the critical value p, and it is shown that the corresponding matings define the same rational map f.

$$< 1/2 < \Theta_1 < 7/12 < \Theta_{n-1} < \Theta_{n-3} < \ldots < \Theta_6 < \Theta_4 < 2/3 < 1 \ ,$$

since the derivative of the real polynomial $f_c(z) = z^2 + c$ is negative for $z < 0$ and then f_c swaps the lower and upper half-planes. Reading off binary digits gives $\Theta_1 = .\overline{10\,01\,01\ldots01\,0}$, which is the largest n-periodic angle less than $7/12 = .10\,\overline{01}$. Reversing these arguments, it follows that the center defined by $\gamma_M(\Theta_1)$ is real and the orbit is as given by (1). Each z_i has two external angles, Θ_i and $1 - \Theta_i$. Note that $f_c : [0, \beta_c] \to [c, \beta_c]$ is increasing; taking preimages of z_i with respect to this branch gives strictly preperiodic points except for z_3, which has a periodic preimage z_2 on the positive half-axis.

Now consider any parameter p in the limb with rotation number k/n, $k = (n-1)/2$. The wake is bounded by $0 < \theta_- < \theta_+ < 1/3$. We have $\theta_+ = .\overline{01\,01\ldots01\,0}$, since this is the largest n-periodic angle less than $1/3 = .\overline{01}$, or by sketching an n-Rabbit. So $\theta_+ = \Theta_3$ and $\theta_- = \Theta_5$; note that $\Theta_1 = 1/2 + \theta_+/4$ is an instance of the Douady Magic Formula from Proposition 2.1. — The critical value p of $f_p(z) = z^2 + p$ is in the sector at α_p bounded by the dynamic rays with the angles $\theta_\pm$. This sector is mapped injectively for $n-1$ iterations; the image sector contains 0, $-\alpha_p$, and a unique point in $f_p^{-1}(-\alpha_p)$. Thus the original sector contains unique preimages of α_p with preperiods n and $n+1$, respectively. Denote the angles of the latter by $\theta_1 < \ldots < \theta_n$. Under n iterations, these are mapped to angles at $-\alpha_p$, such that θ_1 gives the smallest angle in $[1/2, 1]$ and θ_n gives the largest angle in $[0, 1/2]$. So under $n+1$ iterations, θ_1 is mapped to $\theta_+ = \Theta_3$ and θ_n is mapped to $\theta_- = \Theta_5$.

Next, let us look at the Airplane Julia set $\mathcal{K}_q$ with $Q(z) = f_q(z) = z^2 + q$. As the parameter was shifted from c to q, the n-periodic points with angles Θ_i moved holomorphically; in particular the pre-characteristic points corresponding to $\pm z_n$ bound an interval containing the real slice of the Airplane Fatou component around 0. Consider the Fatou component of f_c at z_3; it defines an interval in $\mathcal{K}_q$, which contains a unique preperiodic component Ω of preperiod $n-3$. Its largest antenna in the upper halfplane has angles in a subset of $[\Theta_5\,,\,\Theta_3] = [\theta_-\,,\,\theta_+]$. Since f_q^{n-3} maps it to the largest antenna on the upper side of the Fatou component around 0, f_q^{n-2} maps it behind the component around q. Then it is behind the component around $f_q(q)$, then to the right of the component at 0, and finally we see that f_q^{n+1} maps the antenna of Ω to the interval $(\gamma_q(4/7),\,\beta_q]$. Denote by x_i the preimage of the n-periodic point with angle Θ_i, then x_3 has preperiod n and the others have preperiod $n+1$. On the other hand, the angles θ_i are the only angles of preperiod $n+1$ in $(\theta_-\,,\,\theta_+)$ that are iterated to some Θ_j. Recalling that θ_1 is iterated to $\theta_+ = \Theta_3$, we see that each θ_i with $i \neq 1$ lands at some x_j with $j \neq 3$. Denote the other angle by $\theta_2'\,,\,\dots,\,\theta_n'$; it is in $(\theta_-\,,\,\theta_+)$ as well, since the antenna is contained in an open half-strip bounded by these rays and a real interval.

Finally, define the Misiurewicz parameters $p_* = \gamma_M(\theta_1) = \dots = \gamma_M(\theta_n)$ and $p_i = \gamma_M(\theta_i')$, $i = 2,\,\dots,\,n$. Now p_* is of α-type by construction, so it has preperiod $n+1$ and period 1. The p_i are endpoints, since there is no other hyperbolic component of period n in the k/n-limb; they are pairwise different in particular. Note that for $i = 2,\,\dots,\,n$, the rays $\mathcal{R}_q(-\theta_i')$ and $\mathcal{R}_q(-\theta_i)$ land together as well and the landing point never returns to this wake, so the two rays are homotopic with respect to its orbit and to the real orbit of q, and the encaptures are equivalent: by Proposition 3.1, the matings $Q \amalg P_i \cong Q \amalg P_*$ agree, as do $P_i \amalg Q \cong P_* \amalg Q$. — For the example of $k/n = 2/5$, Figure 2 shows the rays with angles $-\theta_i\,,\,-\theta_i'$ landing pairwise at $\partial\mathcal{K}_q$, and the rays with angles $\theta_i\,,\,\theta_i'$ landing at $\partial\mathcal{K}_{p_*}$, at a preimage of α_{p_*} and at endpoints, respectively.

The landing pattern at $\partial\mathcal{K}_q$ is stable for parameters q between c of period n as above and the Airplane, but the relevant antenna will bifurcate when q is too far behind the Airplane. ■

Note that we have constructed n different matings giving the same rational map, but in contrast to Theorem 4.2, no upper bound on the multiplicity is known in this case. — Assuming that the map $\mathcal{M}_{k/n} \to V_3\,,\ P \mapsto f \cong P \amalg Q$ is continuous, there will be self-intersections of the image corresponding to these shared matings.

6 Examples of mating discontinuity

Geometric mating is not jointly continuous on the subset of $\mathcal{M} \times \mathcal{M}$ where it can be defined. The first three examples below are due to Epstein [11, 5]. Note that all of these techniques involve neutral parameters, and that they do not exclude separate continuity. For specific one-dimensional slices with Q fixed, partial results on continuity have been obtained by Dudko [10] and by Ma Liangang [22].
Rewrite with three subsections: first, DMF and bitransitive, shared old and new. — *Special thanks to Adam Epstein for explaining unpublished results.*

- Let f_λ be a quadratic polynomial with a fixed point of attracting multiplier λ. For $|\lambda| < 1$, $|\mu| < 1$ there are explicit rational maps $F_{\lambda,\mu} \simeq f_\lambda \amalg f_\mu$. Suppose $\lambda,\,\mu \to 1$ tangentially, such that the third multiplier ν is constant. Then if $F_{\lambda,\mu}$ converges to a quadratic rational map, it will depend on ν, so there are oscillating sequences as well. Note that convergence may depend on a normalization allowing the collision of the respective fixed points; in a different normalization, $F_{\lambda,\mu}$ might converge to a map of degree one or to a constant as well.

- Results on shared matings with cluster cycles by Sharland [35, 36] are reported in Section 3. For rotation number $1/n$, we have $f_n \cong R_n \coprod Q_n \simeq P_n \coprod R_n$, where the center parameters correspond to the following roots: $r_n \sim \gamma_M(1/(2^n-1)) = \gamma_M(2/(2^n-1))$, $q_n \sim \gamma_M(-3/(2^n-1)) = \gamma_M(-4/(2^n-1))$, and $p_n \sim \gamma_M((2^{n-1}-1)/(2^n-1)) = \gamma_M(2^{n-1}/(2^n-1))$. Then $r_n \to r_0 = 1/4 = \gamma_M(0)$, $q_n \to q_0 = 1/4 = \gamma_M(0)$, and $p_n \to p_0 = -2 = \gamma_M(1/2)$. Now if mating was continuous, we should have $R_0 \coprod Q_0 \simeq P_0 \coprod R_0$; both geometric matings exist, the former has two parabolic basins and the latter has one.

- For a parabolic or bounded-type Siegel parameter p on the boundary of the main cardioid with angle θ and the real parameter q defined by the Douady Magic Formula $\Theta = 1/2 + \theta/4$ according to Proposition 2.1, consider the geometric mating $f_\theta \cong P \coprod Q$, which exists according tp Blé-Valdez [2, 3]. When θ is irrational, then $f_\theta^2(\infty) = 0$, since the corresponding point in $\mathcal{K}_q$ has the angles $\pm 2\Theta = \pm\theta/2$ and the critical point of P has $\theta/2$ as well. But when θ is rational, then either 0 is in a parabolic basin and ∞ is preperiodic, or there are disjoint cycles of parabolic basins; in both cases $f_\theta^2(\infty) \neq 0$. So approximating a rational angle with irrational ones gives a contradiction to continuity.

- Theorem 6.1 below uses similar ideas to show that the limit is different from the expected one; since only rational angles are used, no special arguments are needed to show matability. Here both p_n and q_n are Misiurewicz polynomials; a concrete example is given below as well.

- Shared matings according to Theorem 4.2 can be used to produce several counterexamples to continuity; here p_n is hyperbolic and q_n is Misiurewicz. Again, the contradiction comes either from a different number of parabolic Fatou cycles, or from an expected limit outside of the Chebyshev family.

- Different kinds of discontinuity may be expected in higher degrees. E.g., with cubic polynomials $f_a(z) = z^3 + az^2$, the mating $f_a \coprod f_{-\bar a}$ gives an antipode-preserving rational map [4]. The former bifurcation locus shall be locally connected at parabolic parameters, while the latter is not. So for suitable sequences of postcritically finite polynomials, there will be an oscillatory behavior.

Theorem 6.1 (Discontinuity with bitransitive family)
Consider a sequence of rational angles $\theta_n \to \theta_0$, such that θ_n and $2\theta_n$ are preperiodic for $n \geq 1$, $2\theta_0$ is periodic, and θ_0 may be either unless θ_0 and $2\theta_0$ belong to the same root. Set $p_n = \gamma_M(\theta_n)$ and $q_n = \gamma_M(-2\theta_n)$ for $n \geq 0$. Then the sequence of geometric matings $f_n \cong P_n \coprod Q_n$ does not converge to $f_0 \cong P_0 \coprod Q_0$.

Proof: First, note that θ and 2θ are never in the same limb, unless both are angles of the root. Thus all geometric matings under consideration exist. Since the angle θ_n of $p_n \in \mathcal{K}_{p_n}$ is complex conjugate to an angle $-\theta_n$ of $0 \in \mathcal{K}_{q_n}$, there is a direct ray connection between these two points, and the rational map satisfies $f_n(0) = \infty$. We have $f_n \not\to f_0$ since $f_0(0) \neq \infty$: while $z = \infty$ has an infinite orbit converging to a parabolic cycle of f_0, $z = 0$ either has a finite orbit or it converges to a different parabolic cycle. — This phenomenon seems to be analogous to parabolic implosion, if we are looking at the polynomials Q_n or at encaptures according to Proposition **??**: $q_n = \gamma_{q_n}(-2\theta_n)$ converges to the critical value q_0 inside a parabolic Fatou component of Q_0, but $\gamma_{q_0}(-2\theta_0)$ is a boundary point of this component. Of course, parabolic implosion looks different for the rational maps here, since the Julia set of f_n is all of $\widehat{\mathbb{C}}$. ■

A concrete example is given by $\theta_n = u_n/2^{2n}$ with $u_n = (2^{2n-1} + 1)/3$. Then p_n and q_n are β-type Misiurewicz points, converging to the Misiurewicz point $p_0 = \mathrm{i} = \gamma_M(1/6)$ and the root $q_0 = -3/4 = \gamma_M(1/3)$, respectively, and the matings do not converge to the mating of the limits. Probably we have a parabolic 2-cycle in both cases, and Fatou components corresponding to a fat Basilica, but the limit of the matings has 0 and ∞ in different components of the Fatou set, while the mating of the limits has 0 in the Julia set at a preimage of the parabolic fixed point.

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