# Convergence of the Thurston Algorithm for quadratic matings

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**Abstract**

Mating is an operation to construct a rational map $f$ from two polynomials, which are not in conjugate limbs of the Mandelbrot set. When the Thurston Algorithm for the unmodified formal mating is iterated in the case of postcritical identifications, it will diverge to the boundary of Teichmüller space, because marked points collide. Here it is shown that the colliding points converge to postcritical points of $f$, and the associated sequence of rational maps converges to $f$ as well. So to compute $f$, it is not necessary to encode the topology of postcritical ray-equivalence classes for the modified mating, but it is enough to implement the pullback map for the formal mating. Different arguments are needed when $f$ is of type (2, 2, 2, 2).

The proof combines local estimates and the Selinger extension of the Thurston Algorithm to augmented Teichmüller space. The latter is illustrated with several examples of canonical obstructions and canonical strata, including a relation between eigenvalues from matings of conjugate polynomials, and their core entropy.

## 1 Introduction

A postcritically finite quadratic polynomial $f_c(z) = z^2+c$ may be periodic of satellite type, periodic of primitive type, or critically preperiodic (Misiurewicz type). Examples are given by the Basilica $Q(z) = z^2 - 1$, the Kokopelli, and by $P(z) = z^2 + \mathrm{i}$ in Figures 1 and 3. Quadratic rational maps have two critical orbits and form a two-parameter family. The dynamics and topology of certain rational maps are understood in terms of one or two polynomials [46]. The operation of mating was introduced by Douady–Hubbard [13]: a rational map $f$ may be described by gluing the Julia sets of $P$ and $Q$, such that points with conjugate external angles are identified. According to Rees–Shishikura–Tan [54, 52], this construction works when $P$ and $Q$ are not in conjugate limbs of the Mandelbrot set: first define the formal mating, where the two Julia sets are in separate half-spheres. The Thurston Theorem [14, 22] shows that there is an equivalent rational map $f$. Then the topological mating is given by collapsing all ray-equivalence classes of the formal mating, and it is conjugate to $f$, so $f$ is a geometric mating. Actually, an intermediate step is required when postcritical points are identified in the mating: then the formal mating will be obstructed, and an unobstructed essential mating is constructed by collapsing a finite number of ray-equivalence classes.

The Thurston Algorithm is based on an iteration in Teichmüller space, which consists of isotopy classes of homeomorphisms. These may be represented by spiders, medusas, or triangulations. Bartholdi–Nekrashevych [2] and Buff-Chéritat [10] employ a path in moduli space instead. Using this "slow" approach, the algorithm shall be faster, easier to implement, and more stable. The slow mating algorithm is related to equipotential gluing in [11]. Figure 1 shows a few snapshots of this process.

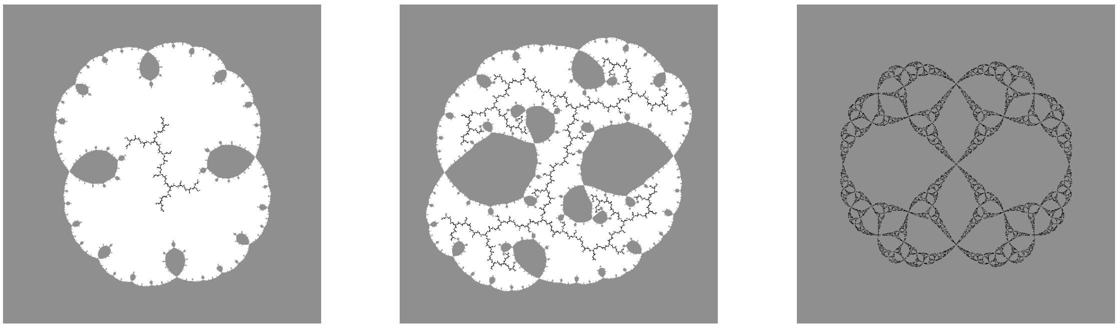

**Figure 1:** Various stages of slow mating, illustrated by moving images $\psi_t(\varphi_0(\mathcal{K}_p))$ and $\psi_t(\varphi_\infty(\mathcal{K}_q))$ of Julia sets. Here $P(z) = z^2 + \mathrm{i}$ is a Misiurewicz polynomial and $Q(z) = z^2 - 1$ is the Basilica polynomial, which has an attracting 2-cycle. The formulas for pulling back marked points and rational maps are discussed in Example 3.5.

In this example, the Thurston Algorithm does not work directly, because the postcritical 2-cycle of $P$ needs to be identified with a fixed point of $Q$: these are connected by external rays, and the ray-equivalence class is surrounded by a removable Thurston obstruction. The classical approach is to construct an essential mating, where certain ray-equivalence classes are collapsed by definition, and to employ the Thurston Algorithm for the modified map. An alternative approach is suggested here: the divergence of the Thurston Algorithm has been described by Nikita Selinger [48, 49] in terms of the augmented Teichmüller space. Applying his characterization to the Thurston Algorithm of the unmodified formal mating, it is shown that marked points come together automatically in the expected way, and the rational maps converge to the geometric mating, at least if $f$ is not of exceptional type (2, 2, 2, 2). The same argument gives convergence of slow mating and equipotential gluing as well, where no modification is appropriate. Thus it is possible to obtain matings numerically without encoding the topology of ray-equivalence classes. In a few more applications, additional obstructions are created and used to prove convergence properties [27, 29]. Here obstructions do not appear as a potential problem, but they are turned into an ally: a powerful tool to show convergence.

The classical Thurston Theorem is discussed in Section 2. See Section 2.6 for examples of canonical obstructions and stabilization of noded Riemann surfaces, including a relation between core entropy and matings of conjugate polynomials. In Section 3, a general convergence result is obtained for bicritical maps in a suitable normalization. The different concepts of mating are developed systematically in Section 4.2, and convergence of mating is discussed in Section 4.3. Various numerical algorithms are compared briefly in Section 5.

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# 2 The Thurston Algorithm

The Thurston Theorem 2.5 gives a combinatorial characterization of branched covers equivalent to rational maps, which is used to describe and to define rational maps, and to construct them numerically: the related Thurston Algorithm provides a convergent sequence of rational maps. An underlying iteration in Teichmüller space $\mathcal{T}$ is needed both to define a unique pullback, and to have analytic tools providing global convergence to a unique fixed point in $\mathcal{T}$. This fails in the presence of Thurston obstructions: then certain annuli get big, curves get short, marked points collide. The process is understood by extending the pullback map to augmented Teichmüller space $\widehat{\mathcal{T}}$.

## 2.1 Hyperbolic geometry and Teichmüller space

A hyperbolic Riemann surface of finite type and genus 0 is isomorphic to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $n \geq 3$ punctures. Although the manifold extends analytically to a puncture or **marked point**, the hyperbolic metric is infinite there. We shall deal with homotopy classes of simple closed curves and the hyperbolic length of geodesics:

- A simple closed curve in the complement of the marked points is **essential**, if each disk in the complement of the curve contains at least two marked points;
- **peripheral**, if one component contains only one marked point; and
- **trivial** or null-homotopic, if one component contains no marked point.

Note that some authors say non-peripheral instead of essential, or inessential instead of peripheral. The following properties of hyperbolic geodesics are fundamental:

**Proposition 2.1 (Hyperbolic geodesics)**
*Consider the hyperbolic metric on $\widehat{\mathbb{C}}$ with $n \geq 3$ punctures:*

1. *For any essential simple closed curve there is a unique geodesic homotopic to it.*

2. *A simple closed geodesic $\gamma$ has a **collar** neighborhood, an embedded annulus of definite width. Collars around disjoint geodesics are disjoint, and a geodesic crossing the collar of $\gamma$ has an explicit lower bound on its length, which goes to $\infty$ when $l(\gamma) \to 0$. In particular, all sufficiently short geodesics are disjoint.*

3. *Any annulus around $\gamma$ has modulus bounded above by $\pi/l(\gamma)$. The collar has modulus bounded below by $\pi/l(\gamma) - 1$.*

4. *For a sequence of surfaces in a suitable normalization, two marked points collide with respect to the spherical metric, if and only if a hyperbolic geodesic separating them from two other marked points has length going to* 0.

References for the **proof:** See [21] for item 1 and [14, 21, 9] for items 2 and 3. Item 4 is a standard estimate for extremal annuli. ∎

For an implicit $n \geq 3$, **moduli space** $\mathcal{M}$ is the space of Riemann spheres with $n$ marked points, up to Möbius maps or normalization of three points; in our case of genus 0, it has an explicit description as a subset of $\widehat{\mathbb{C}}^{n-3}$ from the positions of marked points. Now **Teichmüller space** $\mathcal{T}$ is the universal cover of $\mathcal{M}$. It can be described by isotopy classes of orientation-preserving homeomorphisms $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ between spheres with marked points; here the left sphere is a topological sphere fixed for reference. Although it has no complex structure, let us write $\widehat{\mathbb{C}}$ instead of $S^2$ nevertheless: this allows to use explicit coordinates and formulas from $\mathbb{C}$. The **projection** $\pi : \mathcal{T} \to \mathcal{M}$ gives the universal cover, and the **pure mapping class group** $G$ is the group of deck transformations: $[h] \in G$ is an isotopy class of homeomorphisms of the topological sphere fixing the marked points, which acts on $\mathcal{T}$ by $[h] \cdot [\psi] = [\psi \circ h^{-1}]$; $G$ is generated by Dehn twists [21].

There are various metrics on $\mathcal{T}$, such that $G$ acts by isometries and the metrics project to $\mathcal{M}$. Actually the definition as Finsler metrics is lifted from $\mathcal{M}$ to $\mathcal{T}$ locally. The dual tangent space is given by integrable quadratic differentials, and by using different norms there, the infinitesimal metrics $\|\mathrm{d}\tau\|_T$ and $\|\mathrm{d}\tau\|_{WP}$ are obtained; integration along shortest curves defines the **Teichmüller metric** $d_T$ and the **Weil–Petersson metric** $d_{WP}$ [21, 58, 33]. The Teichmüller metric is given equivalently by $d_T([\psi_1], [\psi_2]) = 1/2 \inf \log K(\psi)$, where $\psi$ is isotopic to $\psi_1 \circ \psi_2^{-1}$ and $K$ denotes the quasi-conformal dilatation.

**Proposition 2.2 (Basic properties of Teichmüller space)**
1. *$\mathcal{T}$ and $\mathcal{M}$ are analytic manifolds and the universal cover $\pi : \mathcal{T} \to \mathcal{M}$ is analytic. It is a local isometry for both metrics, $d_T$ and $d_{WP}$, and $G$ acts by isometries.*

2. *Both metrics generate the same topology on $\mathcal{T}$. Now $\mathcal{T}$ is not compact, is complete with respect to $d_T$, and incomplete with respect to $d_{WP}$.*

3. *For an essential curve $\gamma$ in the topological sphere and $\tau \in \mathcal{T}$, denote by $l(\gamma, \tau)$ the **length** of the geodesic in the Riemann surface $\pi(\tau)$, that is homotopic to $\psi(\gamma)$ for $\psi \in \tau$. This length is continuous on $\mathcal{T}$ with $|\log l(\gamma, \tau') - \log l(\gamma, \tau)| \leq 2d_T(\tau', \tau)$.*

4. *There are finitely many essential curves $\gamma_i$, such that the collection of length functions $l(\gamma_i, \tau)$ determines $\tau$ uniquely.*

5. *There is a relative estimate $\|\mathrm{d}\tau\|_{WP} \leq C_*\|\mathrm{d}\tau\|_T$ with $C_* = C_*(\mathcal{T})$.*

6. *For $R > 0$ there is $D_* = D_*(R, \mathcal{T}) > 0$ such that all $\tau$ with shortest geodesic length $l_*(\tau) \geq R$ satisfy: all $\tau'$ with $d_{WP}(\tau', \tau) \leq D_*$ have $d_T(\tau', \tau) \leq 1/4$.*

7. *A closed subset of $\mathcal{M}$ is compact, if and only if the length of all simple closed geodesics is bounded uniformly below.*

References for the **proof:** See [21, 58, 33] for items 1–4 and [35] for item 5.
6. According to Lemma 3.22 in [33] we have the relative estimate

$$\|\mathrm{d}\tau\|_T \leq \frac{C}{l_*(\tau)} \|\mathrm{d}\tau\|_{WP} \, , \tag{1}$$

where $C$ depends only on the Teichmüller space $\mathcal{T}$ and $l_*$ denotes the length of the shortest hyperbolic geodesic on the Riemann surface $\pi(\tau)$. So we must choose $D_*$ small enough to have a lower bound for $l_*(\cdot)$ on the $WP$-geodesic from $\tau'$ to $\tau$. Note that it is not sufficient to have a lower bound at $\tau'$ and $\tau$ only; cf. Remark 2.9. Suppose $\tau'$ has $l_*(\tau') < R/2$, then for $\varepsilon > 0$ there is a subarc $[\tilde{\tau}', \tilde{\tau}]$ of the $WP$-geodesic $[\tau', \tau]$ and a simple closed curve $\gamma$, such that $l(\gamma, \cdot)$ increases from $R/2+\varepsilon$ to $R$ along this subarc while $l_*(\cdot) \geq R/2$. Now $|\mathrm{d}l| \leq 2l\|\mathrm{d}\tau\|_T$ and (1) give

$$d_{WP}(\tau', \tau) > d_{WP}(\tilde{\tau}', \tilde{\tau}) \geq \frac{1}{2C}\int_{[\tilde{\tau}', \tilde{\tau}]} \frac{l_*(\cdot)}{l(\gamma, \cdot)}\mathrm{d}l(\gamma, \cdot) \geq \frac{1}{4C}\int_{R/2+\varepsilon}^{R} \mathrm{d}l \;, \tag{2}$$

so if $D_* = \frac{R}{8C}$ and $d_{WP}(\tau', \tau) \leq D_*$ then $l_*(\cdot) \geq R/2$ on the $WP$-geodesic $[\tau', \tau]$ and (1) gives $d_T(\tau', \tau) \leq \frac{2C}{R} \cdot D_* = 1/4$.
7. This is the Mumford compactness theorem, whose proof is simplified in genus 0: the length is bounded below on any bounded ball. Every sequence in $\mathcal{M}$ has a convergent subsequence in $\widehat{\mathbb{C}}^{n-3}$, but if the length of geodesics may go to 0, marked points collide according to Proposition 2.1.4 and the limit does not belong to $\mathcal{M}$. Note that every compact subset of $\mathcal{T}$ has length bounded below as well, but the converse is wrong: for general $g \in G$ the sequence $\tau_k = g^k \cdot \tau_0$ does not accumulate in $\mathcal{T}$, but the collection $\{l(\gamma, \tau_k)\}$ is independent of $k$. ∎

## 2.2 Thurston maps and pullback map

A postcritically finite rational map $f$ is not characterized uniquely by its ramification portrait. Except for flexible Lattès maps, the additional topological information can be given combinatorially. For polynomials, Hubbard trees and external angles provide an explicit description. For rational maps, the combinatorial object is an equivalence class of Thurston maps:

- A **Thurston map** $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an orientation-preserving branched cover of degree $d \geq 2$ with finite postcritical set $P$ and marked set $Z \supset P$. Here $P$ contains all forward iterates of critical points and $Z$ may contain additional critical, preperiodic, and periodic points, such that $g(Z) \subset Z$.

- Two Thurston maps $f$, $g$ are Thurston equivalent or **combinatorially equivalent**, if there are homeomorphisms $\psi_0$, $\psi_1$ with $\psi_0 \circ g = f \circ \psi_1$, $\psi_0 = \psi_1$ on $Z_g$, $\psi_i(Z_g) = Z_f$, and $\psi_1$ is isotopic to $\psi_0$ relative to $Z_g$. So $g$ is deformed continuously to $\psi_1^{-1} \circ f \circ \psi_1$.

- A pullback map is associated with each Thurston map $g$ as follows: for any homeomorphism $\psi$ there is a rational map $f$ and another homeomorphism $\psi'$ with $\psi \circ g = f \circ \psi'$; see [14, 22, 9]. The complex structure defined by $\psi$ is pulled back with $g$ and integrated with $\psi'$. These functions are unique up to Möbius maps, or unique after normalizing three marked points to $\infty$, 0, 1. It turns out that the isotopy class of $\psi$ determines the isotopy class of $\psi'$, so an analytic **pullback map** $\sigma_g : \mathcal{T} \to \mathcal{T}$ is defined by $\sigma_g([\psi]) = [\psi']$.

- The fixed points of $\sigma_g$ in $\mathcal{T}$ correspond to Möbius conjugacy classes of rational maps. Under suitable conditions, the pullback map will be strictly contracting, and the **Thurston Algorithm** converges in addition: define $\psi_n \circ g = f_n \circ \psi_{n+1}$ recursively, then $f_n \to f$ and $[\psi_n]$ converges in $\mathcal{T}$ to the fixed point of $\sigma_g$.

The ramification portrait of $g$ translates to relations between the images of marked points, $x_i = \psi(z_i)$ and $x'_i = \psi'(z_i)$, such that $g(z_i) = z_j$ implies $f(x'_i) = x_j$ when $\psi \circ g = f \circ \psi'$. In the bicritical case with marked critical points, $f$ is determined by $x_i$ and the $x'_i$ are obtained up to the branch of the $d$-th root. See Examples 3.5 and 3.6. Note that we cannot pull back marked points with the resulting formulas alone, since the choice of branch is determined by the pullback in Teichmüller space. The following proposition from [14, 22, 9, 48] gives contraction properties of $\sigma_g$; see Section 2.3 for the relation between convergence and Thurston obstructions, and Remark 2.4 for the notation (2, 2, 2, 2).

**Proposition 2.3 (Thurston–Selinger)**
*Consider a Thurston map $g$ of degree $d \geq 2$ and the pullback map $\sigma_g$.*

1. *$\sigma_g$ is weakly contracting with respect to the Teichmüller metric.*
2. *If $g$ has not type (2, 2, 2, 2), then some iterate of $\sigma_g$ is strictly contracting. The contraction is uniform on subsets of $\mathcal{T}$, such that $\pi(\tau)$ varies in a compact subset of $\mathcal{M}$.*
3. *$\sigma_g$ is Lipschitz continuous on $\mathcal{T}$ with respect to the Weil-Petersson metric; a factor is given by $\sqrt{d}$.*

## 2.3 Obstructions and the Thurston Theorem

A **multicurve** is a nonempty collection of pairwise disjoint and non-homotopic essential curves. The preimage of a multicurve $\Gamma$ under a Thurston map $g$ is $\Gamma' \cup \Gamma''$, where $\Gamma'$ consists of essential curves and the curves in $\Gamma''$ are peripheral or trivial. The homotopy class of $\Gamma'$ depends only on the homotopy classes of $\Gamma$ and on $g$. Note that $\Gamma'$ may contain mutually homotopic curves and it may be empty as well. $\Gamma$ is called invariant, if every curve in $\Gamma'$ is homotopic to a curve in $\Gamma$; it is completely invariant if the converse holds in addition.

For $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ the Thurston matrix $M_\Gamma = (m_{ij})$ is defined as $m_{ij} = \sum 1/d_{ijk}$, where the sum runs over all preimages of $\gamma_j$ homotopic to $\gamma_i$ and $d_{ijk}$ is the degree of $g$ on these preimages. Now $\Gamma$ is a Thurston **obstruction**, if the leading eigenvalue $\lambda_\Gamma$ of $M_\Gamma$ satisfies $\lambda_\Gamma \geq 1$. There are different conventions, whether invariance is required. Most important are the following kinds of obstructions:

- An obstruction $\Gamma$ is a **simple obstruction**, if no permutation turns $M_\Gamma$ into a lower-triangular block form, such that the upper left block has leading eigenvalue $< 1$. A simple obstruction is always completely invariant.
- A multicurve $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ is a **Lévy cycle**, if each $\gamma_i$ is homotopic to a preimage of $\gamma_{i+1 \, mod \, n}$ and the corresponding degree is one. Then $\Gamma$ need not be invariant, but it can be extended to a simple obstruction. The converse holds in the quadratic or bicritical case: every simple obstruction contains a Lévy cycle. These are classified further in [54, 46].

Obstructions are important for the Thurston pullback, because they are related to the presence of annuli with large modulus and of short geodesics [14, 22, 9]; see also Section 2.5 for more explicit statements.

**Remark 2.4 (Exceptional case of Thurston maps)**
*The Thurston Algorithm requires special treatment of the following maps: a Thurston map $g$ of type $(2, 2, 2, 2)$ has four postcritical points, the critical points are non-degenerate and not postcritical. The notation $(2, 2, 2, 2)$ refers to an orbifold as explained in [41, 34], which is not needed in the present paper. Although formal matings from non-conjugate limbs are never of type $(2, 2, 2, 2)$, the essential mating may be [54, 40, 27].*

**Theorem 2.5 (Thurston–Pilgrim, general case)**
*Suppose $g$ is a Thurston map of degree $d \geq 2$, not of type $(2, 2, 2, 2)$, possibly with additional marked points. Then:*
- *Either there is no Thurston obstruction, $g$ is combinatorially equivalent to a rational map $f$, which is unique up to Möbius conjugation, and the Thurston pullback $\sigma_g$ converges globally to its unique fixed point.*
- *Or $g$ is obstructed, it is not equivalent to a rational map, and the Thurston pullback diverges. There is a unique* ***canonical obstruction*** *$\Gamma$, such that every $\gamma \in \Gamma$ is pinched, while the length of every other curve is bounded uniformly from below.*

See [14, 22, 9, 44] for the original **proof**. An alternative proof by Selinger [48] is given in Section 2.5. Note that the fine print reads as follows: if there is an obstruction, it need not be pinching, but then it implies the existence of another obstruction, which is pinching. The pinching obstructions show that marked points get identified in the limit, which means divergence to the boundary in Teichmüller space and in moduli space as well. Then the rational maps may still converge to a rational map of degree $d$, which is not equivalent to $g$, or to a map of lower degree. Note that there is no algorithm to find obstructions in a time bounded a priori.

Originally this theorem was stated under the assumption of a hyperbolic orbifold [14, 22]. A parabolic orbifold is either of type $(2, 2, 2, 2)$ as defined above, or there are only two or three postcritical points. In that case, the Thurston pullback is undefined or constant, respectively, unless there are additional marked points — then the pullback map is strictly contracting as in the hyperbolic orbifold case [9].

For quadratic Thurston maps of type $(2, 2, 2, 2)$, some things are the same, some are different: a fixed point of $\sigma_g$ is still unique, when it exists, but it is not attracting. Every obstruction is pinching, and it excludes a fixed point, but there are unobstructed maps not equivalent to a rational map as well. The converse happens when the degree $d \geq 4$ is a square: there is a one-parameter family of flexible Lattès maps, which are mutually equivalent but not Möbius conjugate. So uniqueness fails, and moreover, there is a non-pinching obstruction. For Thurston maps of type $(2, 2, 2, 2)$ with additional marked points, pinching and non-pinching obstructions are characterized by Selinger–Yampolsky [49, 50].

Dylan P. Thurston [55] obtains a positive criterion for $g$ to be equivalent to a rational map, at least if there is a periodic critical point: $f^{-k}$ is uniformly contracting on a graph, which forms a spine for $\widehat{\mathbb{C}} \setminus P$. In [45], Kevin Pilgrim gives an algebraic characterization of obstructions by non-contraction of the virtual endomorphism of the pure mapping class group. See [2, 4] for algebraic descriptions of Thurston maps in terms of iterated monodromy groups or bisets.

## 2.4 Augmented Teichmüller space and moduli space

When a Thurston map $g$ is not combinatorially equivalent to a rational map, so $\sigma_g$ has no fixed point in $\mathcal{T}$, we may understand this by considering a space larger than $\mathcal{T}$ or with a different topology. Except for type (2, 2, 2, 2), divergence of the Thurston Algorithm is related to collisions of marked points and pinching of essential curves.

- **Augmented moduli space** $\widehat{\mathcal{M}}$ describes noded Riemann surfaces up to conformal equivalence. These surfaces are unions of spheres $\widehat{\mathbb{C}}$ with marked points and common nodes. Each sphere has at least three marked points and nodes, and in genus 0, the spheres form a tree. There are only finitely many boundary strata $\mathcal{S}_{G\cdot\Gamma} \subset \widehat{\mathcal{M}}$.

- **Augmented Teichmüller space** $\widehat{\mathcal{T}}$ has classes of continuous maps from a topological sphere to a possibly noded Riemann surface, which may send certain curves to single points. These maps are equivalent under an isotopy in the domain or a product of Möbius transformations in the range. Boundary strata $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$ are labeled by homotopy classes of pinched multicurves. As in the case of $\mathcal{T}$ and $\mathcal{M}$, non-homotopic curves in the topological sphere may be mapped to the same short geodesic by different maps — this defines different elements of $\widehat{\mathcal{T}}$ but the same element of $\widehat{\mathcal{M}}$. Boundary strata of $\widehat{\mathcal{T}}$ are products of lower-dimensional Teichmüller spaces.

- A neighborhood base for the topology of $\widehat{\mathcal{T}}$ or $\widehat{\mathcal{M}}$ is defined in terms of maps between noded Riemann surfaces, which map $\varepsilon$-short geodesics to nodes and which are $(1+\varepsilon)$-quasiconformal outside of the collars. Using both Fenchel–Nielsen coordinates [21] and plumbing coordinates, which take Example 2.6 as a local model, the "infinitely branched" cover $\pi : \widehat{\mathcal{T}} \to \widehat{\mathcal{M}}$ is understood locally. So $\widehat{\mathcal{M}}$ is a compact analytic space [20].

The notion of noded Riemann surfaces is motivated here by pinching obstructions; originally they were introduced to compactify $\mathcal{M}$, and to describe algebraic curves with self-intersections. These constructions are due to Deligne–Mumford, Bers, Abikoff, and Masur; see the references in [20, 58, 33]. The following standard example shows the degeneration of a Riemann surface with boundary explicitly:

**Example 2.6 (Pinching a short geodesic)**
*Consider the Riemann surface* $S_t = \{(x,\, y) \in \mathbb{C}^2 \,|\, |x| < 1,\, |y| < 1,\, x \cdot y = t\}$ *for* $0 < |t| < 1$*. As* $t \to 0$*,* $S_t$ *becomes the union of two disks intersecting transversely in the single point* $(0,\, 0)$*. The hyperbolic metric in the annulus* $|t| < |x| < 1$ *is known explicitly, and seen to converge to the hyperbolic metric of the punctured disk. When we try to illustrate this process in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*, either* $S_t$ *looks disconnected or the limit does not show a transversal intersection of smooth manifolds.*

This example shall motivate that we are interested in surfaces consisting of smooth spheres intersecting transversely. The nodes appear as additional marked points in the pieces, because the hyperbolic metric is singular there. An approximate Riemann surface would have long, thin tunnels between thick components; this is symbolized by connected spheres as in Figure 2.

**Example 2.7 (Augmented moduli space)**
*1. Consider $\widehat{\mathbb{C}}$ with four marked points $x_1 = \infty$, $x_2 = 0$, $x_3 = 1$, $x_4 = a$. The moduli space is given by $a \in \widehat{\mathbb{C}} \setminus \{\infty, 0, 1\}$ and the augmented moduli space is $\widehat{\mathcal{M}} = \widehat{\mathbb{C}}$: e.g., $a \to 0$ corresponds to pinching a curve separating $x_2 = 0$ and $x_4 = a$ from $x_3 = 1$ and $x_1 = \infty$. Note that division by $a$ gives a different normalization $x_1 = \infty$, $x_2 = 0$, $x_3 = 1/a$, $x_4 = 1$; now $a \to 0$ means $x_3 \to x_1$. In fact this is the same Riemann surface as before, with a node separating $x_2$, $x_4$ from $x_3$, $x_1$.*

*2. The case of five marked points is described by $\mathcal{M} = \{(a, b) \in \widehat{\mathbb{C}}^2 \,|\, a, b \neq \infty, 0, 1, a \neq b\}$, but now the topology of $\widehat{\mathcal{M}}$ is more involved than $\widehat{\mathbb{C}}^2$: E.g., one-dimensional boundary strata are given by $a = 0$, $b \neq \infty, 0, 1$ or by $a = b \neq \infty, 0, 1$, but we lose information when $a = b = 0$: this may be one of three 0-dimensional strata, or in a one-dimensional stratum without information on the relative position of three marked points and a node.*

**Proposition 2.8 (Augmented Teichmüller space)**
*$\widehat{\mathcal{T}}$ and $\widehat{\mathcal{M}}$ are topological spaces, such that $\pi : \mathcal{T} \to \mathcal{M}$ extends to a continuous map $\pi : \widehat{\mathcal{T}} \to \widehat{\mathcal{M}}$.*

1. *The Weil–Petersson metric $d_{WP}$ extends to $\widehat{\mathcal{T}}$ and $\widehat{\mathcal{M}}$, such that $\widehat{\mathcal{T}}$ is the completion of $\mathcal{T}$ and $\widehat{\mathcal{M}}$ is a compactification of $\mathcal{M}$. On each boundary stratum, the extended $d_{WP}$ is a product of lower-dimensional Weil–Petersson metrics.*
2. *Each point $\tau \in \widehat{\mathcal{T}}$ is approximated only from finitely many strata.*
3. *Normalizing three marked points, the coordinates of all marked points extend continuously from $\mathcal{M}$ to $\widehat{\mathcal{M}}$ or from $\mathcal{T}$ to $\widehat{\mathcal{T}}$.*
4. *For every essential simple closed curve $\gamma$, the length $l(\gamma, \tau) \in [0, \infty]$ is continuous on $\widehat{\mathcal{T}}$. All length functions together determine $\tau$ uniquely.*

Explanations, references, and sketch of a **proof**: See [20, 58, 33] for item 1. Note that $\widehat{\mathcal{T}}$ is only a partial compactification of $\mathcal{T}$: a sequence leaving $\mathcal{T}$ may converge in $\widehat{\mathcal{T}}$, if esential curves are pinched, but a sequence of the form $g^k \cdot \tau_0$ with a Dehn twist $g \in G$ will diverge in $\widehat{\mathcal{T}}$ as well.

2. This follows from the definition of neighborhoods given above.

3. Continuity is obtained from extending the $(1 + \varepsilon)$-quasiconformal maps into approximately round collars, or from a compactness argument and continuity of length. The normalization singles out a sphere, where all marked points have limits, while marked points in other components converge to nodes of this sphere. The statement is equivalent to a continuous extension of cross-ratios; in [15] a completion with respect to cross-ratios is used to construct a space isomorphic to $\widehat{\mathcal{T}}$.

4. See the references above and [48]. Approaching a lower-dimensional stratum according to item 2, specific curves have length $\to 0$ and intersecting curves have length $\to \infty$. For all other curves, the hyperbolic metric converges on each component in a suitable normalization. Injectivity of length functions follows from Proposition 2.2.4. ∎

The pure mapping class group $G$ acts on $\widehat{\mathcal{T}}$ by Weil–Petersson isometries, but the description of $\widehat{\mathcal{M}} = \widehat{\mathcal{T}}/G$ is more involved:

**Remark 2.9 (Action of $G$ and uniqueness of geodesics)**
*Near a boundary stratum $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$, distinguish the following kinds of Dehn twists $g \in G$ about $\gamma$:*

*a) If $\gamma$ intersects a curve in $\Gamma$, the action of $g$ would map to a different stratum.*

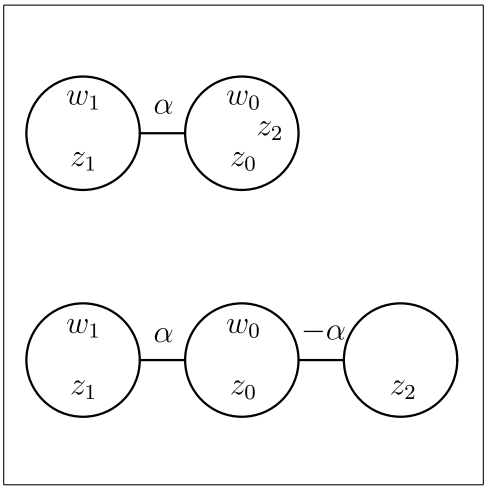
*b) If $\gamma$ is contained in a component of $\widehat{\mathbb{C}} \setminus \Gamma$, then $g$ acts from the pure mapping class group of the component space.*

*c) If $\gamma \in \Gamma$, then $g$ acts trivially on the stratum but not trivially in a neighborhood. For $\tau \in \mathcal{T}$ close to $\mathcal{S}_\Gamma$, $d_{WP}(\tau,\, g^k \cdot \tau)$ is bounded by the triangle inequality, but $d_T(\tau,\, g^k \cdot \tau) \to \infty$, although the length of all hyperbolic geodesics is bounded below; cf. Proposition 2.2.*

*So $\pi$ is "infinitely branched" and not a local isometry. According to [58], $\widehat{\mathcal{T}}$ is a unique geodesic space nevertheless, with open geodesics passing through a unique stratum of lowest possible dimension.*

## 2.5 Extended pullback map and the canonical obstruction

To extend the pullback map $\sigma_g$ to augmented Teichmüller space, consider a multicurve $\Gamma$ and the collection $\Gamma'$ of non-homotopic essential preimages. Then $\sigma_g$ shall map $\mathcal{S}_\Gamma$ to $\mathcal{S}_{\Gamma'}$. The definition is understood by considering the full preimage $g^{-1}(\Gamma)$ first; this defines noded surfaces with possibly non-hyperbolic pieces. So whenever a disk component contains at most one marked point, it is reduced to a point, and an annulus between homotopic curves is reduced to a point as well. This process of stabilization defines a noded Riemann surface with hyperbolic pieces.



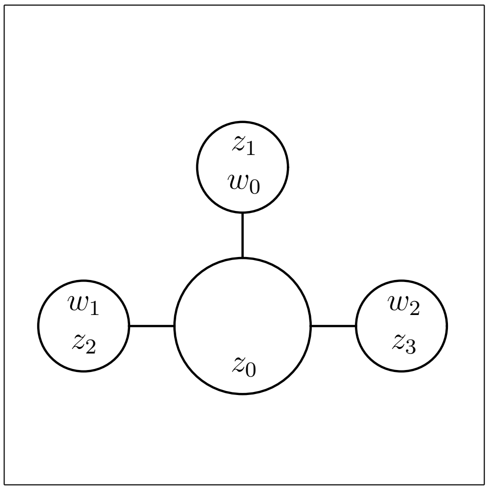




**Figure 2:** Left: The formal mating $g$ of Airplane and Basilica has a cyclic ray connection $\Gamma = \{\gamma\}$ between the two $\alpha$-fixed points, which is the canonical obstruction. The preimage $\Gamma' \cup \Gamma''$ contains a peripheral curve, so the right sphere in the lower surface $\widehat{\mathbb{C}}/\Gamma'$ is considered as one point. Then $g_\Gamma$ is a self-map of the noded surface $\widehat{\mathbb{C}}/\Gamma$ and $z_0$ is no longer 3-periodic $z_0 \Rightarrow z_1 \to z_2 \to z_0$, but preperiodic with $z_0 \Rightarrow z_1 \to z_2 \to \alpha \uparrow$.
Right: The formal mating $1/4 \sqcup 1/2$ with marked critical points has three curves in the canonical obstruction, which surround ray connections with the angles 1/4, 1/2, 0. Due to the identification of points in the small pieces, the geometric mating $f \cong 1/4 \coprod 1/2$ satisfies $f(0) = \infty$.

So when $\sigma_g([\psi]) = [\psi']$ and $\psi$ maps the curves of $\Gamma$ to nodes, then $\psi'$ maps curves in $\Gamma'$ to nodes, and certain annuli and disks to nodes or marked points as well. The pullback map on a product of lower-dimensional Teichmüller spaces is described in terms of homeomorphisms or Thurston maps $g^C$ on the pieces $C$, which are defined uniquely up to combinatorial equivalence. Note that for a completely invariant multicurve $\Gamma$, appropriate identifications must be made to describe a $\sigma_g$-invariant boundary stratum $\mathcal{S}_\Gamma$. The collection $g_\Gamma$ of component maps is defined on the topological model surface $\widehat{\mathbb{C}}/\Gamma$. The examples in Figure 2 show that it may be discontinuous, not surjective, and it may map marked points to nodes.

**Theorem 2.10 (Selinger extension)**
*For a Thurston map $g$ of degree $d \geq 2$, the Thurston pullback $\sigma_g$ has a unique continuous extension to $\widehat{\mathcal{T}}$. On each boundary stratum, it is given by a pullback with component maps as described above.*

**Idea of the proof:** A unique extension is given by completion, using the uniform Lipschitz estimate from Proposition 2.3.3. For the explicit extension above, the length functions of all curves are continuous when a lower-dimensional stratum is approximated from a higher-dimensional stratum [48]. By Proposition 2.8.4, both extensions agree. ■

The following result is due to Pilgrim [44] in the case of a hyperbolic orbifold. The proof by Selinger [48] works for maps of type (2, 2, 2, 2) as well.

**Theorem 2.11 (Canonical obstruction by Pilgrim–Selinger)**
*Suppose $g$ is a Thurston map of degree $d \geq 2$, fix $\tau_0 \in \mathcal{T}$, and set $\tau_n = \sigma_g^n(\tau_0)$. There is an $R(\tau_0) > 0$ and a multicurve $\Gamma$, possibly empty, such that:*
- *If $\gamma \in \Gamma$, then $l(\gamma, \tau_n) \to 0$.*
- *If $\gamma \notin \Gamma$, then $l(\gamma, \tau_n) \geq R(\tau_0)$.*

*In augmented moduli space, $\pi(\tau_n)$ accumulates at a compact subset of the canonical stratum $\mathcal{S}_{G \cdot \Gamma} \subset \widehat{\mathcal{M}}$. Up to homotopy, the **canonical obstruction** $\Gamma$ is independent of $\tau_0$. If $\Gamma \neq \emptyset$, it is a simple Thurston obstruction, and the curves of $\Gamma$ do not intersect another curve from any simple obstruction.*

This implies that every accumulation point of $\tau_n$ belongs to the canonical stratum $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$, but there need not be accumulation in $\widehat{\mathcal{T}}$ at all. The accumulation sets in $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{T}}$ may depend on the starting point $\tau_0$. While multicurves and obstructions are never empty, it is customary to say "$\Gamma$ is empty" instead of "there is no $\Gamma$" here. Recall the definition of the Thurston matrix $M_\Gamma$ from Section 2.3. We have:

- If $\Gamma$ is a simple obstruction, $M_\Gamma$ has a positive eigenvector $v$ with eigenvalue $\lambda_\Gamma \geq 1$. Suppose that in the Riemann surface $\pi(\tau)$, there are annuli around the corresponding geodesics with moduli proportional to $v$, then these moduli will grow at least by $\lambda_\Gamma$ under the pullback $\pi(\sigma_g(\tau))$. This is a direct application of the Grötsch inequality [21]. By the collar estimate from Proposition 2.1.3, a lower bound on the modulus corresponds to an upper bound on the hyperbolic length of the geodesic.

- For sufficiently short geodesics, there is a kind of reverse estimate: when $\Gamma$ is completely invariant but not a simple obstruction, there is a semi-norm on the vector of inverse lengths, which cannot increase arbitrarily. Here a preimage annulus is decomposed along parallels to the core curve, and the new annuli are related to inverse length by the collar theorem again. See Theorem 7.1 in [14], Theorem 10.10.3 in [22], or Lemma 2.6 in [9].

Sketch of the **proof** of Theorem 2.11: Let $N \geq 0$ be the maximal number of arbitrarily short geodesics in $\pi(\tau_n)$ as $n \to \infty$. So there is $R > 0$, a subsequence $n_k$, multicurves $\Gamma_k$ with $N$ elements, and $\varepsilon_k \to 0$, such that:

- For each $n$, there are at most $N$ curves $\gamma$ with $l(\gamma,\, \tau_n) < R$.
- For all $k$ and all $\gamma \in \Gamma_k$ we have $l(\gamma,\, \tau_{n_k}) < \varepsilon_k$.

Now if $N = 0$, the claims are satisfied for $\Gamma = \emptyset$. So assume $N > 0$. For large $k$ we have $\varepsilon_k << R$ and continuity of $l(\gamma, \cdot)$ together with the reverse inequality above shows that $\Gamma_k$ is completely invariant. We may assume that all $\Gamma_k$ have the same partition of marked points $G \cdot \Gamma_k$, the same Thurston matrix $M = M_{\Gamma_k}$, and $\pi(\tau_{n_k})$ has a limit in $\mathcal{S}_{G\cdot\Gamma_k} \subset \widehat{\mathcal{M}}$. If $\Gamma_k$ was not a simple obstruction, the reverse inequality applied to $M$ would give a lower bound for $\varepsilon_k$. This is a contradiction for some large $k$, and we set $\Gamma = \Gamma_k$. Now for all $n \geq n_k$ the moduli of annuli around $\gamma \in \Gamma$ have non-decreasing lower bounds and the lengths $l(\gamma,\, \tau_n)$ have non-increasing upper bounds; together with the assumptions on the subsequence this gives $l(\gamma,\, \tau_n) \to 0$. The lower bound $R$ is satisfied by all other curves, and Mumford compactness according to Proposition 2.2.7 applies to all components of $\mathcal{S}_{G\cdot\Gamma}$. If some $\gamma \in \Gamma$ was intersecting a $\gamma'$ from another simple obstruction homotopically transversely, there would be an upper bound for $l(\gamma',\, \tau_n)$ and a lower bound for $l(\gamma,\, \tau_n)$ by the collar theorem. Finally, for a different initial $\tau_0$ all length $l(\gamma,\, \tau_n)$ are changed by a factor bounded above and below, so $\Gamma$ is independent of $\tau_0$. ∎

An alternative **proof of the Thurston Theorem 2.5** based on Theorem 2.11: Consider $\tau_n = \sigma_g^n(\tau_0)$ for some $\tau_0 \in \mathcal{T}$. If the canonical obstruction is $\Gamma \neq \emptyset$, then $\pi(\tau_n)$ leaves every compact subset of $\mathcal{M}$, so $\sigma_g$ cannot have a fixed point in $\mathcal{T}$ and there is no rational map $f$ equivalent to $g$.

Now assume $\Gamma = \emptyset$. Then $\pi(\tau_n)$ stays in a compact subset of $\mathcal{M}$. If $g$ is of type $(2,\, 2,\, 2,\, 2)$, it may be obstructed or not, equivalent to a rational map or not. But otherwise some iterate of $\sigma_g$ is uniformly contracting over the compact set of $\mathcal{M}$ defined by $R(\tau_0)$. So $\tau_n$ converges to a fixed point, which corresponds to a rational map $f$. If $g$ had a non-canonical simple obstruction $\widetilde{\Gamma}$, then $\tau_n$ could not converge to the fixed point when $\tau_0$ had $l(\gamma,\, \tau_0)$ too small for $\gamma \in \widetilde{\Gamma}$. ∎

## 2.6 Characterization of the canonical obstruction

There is no finite algorithm to determine obstructions of an arbitrary Thurston map $g$, but if you have a guess what the canonical obstruction $\Gamma$ might be, this can be checked with the following criterion. In particular, it shows that the canonical obstruction of a formal mating from non-conjugate limbs is given by loops around ray-equivalence classes with at least two postcritical points; see Section 4.3. Examples of canonical obstructions are given in Figures 2 and 3 as well.

**Theorem 2.12 (Selinger characterization of the canonical obstruction)** *When $g$ is a Thurston map of degree $d \geq 2$, consider the family of multicurves $\widetilde{\Gamma}$, which are simple obstructions or empty, with the following property: for the map $g_{\widetilde{\Gamma}}$ between components of the noded surface defined by $\widetilde{\Gamma}$, the first-return map of each periodic component is*

- *a homeomorphism,*
- *an unobstructed Thurston map, or*

• *a* (2, 2, 2, 2)*-map with a non-pinching obstruction: all curves are essential with respect to the four postcritical points, and the degree of the map is a square.*
*Now up to homotopy, this family of multicurves* $\widetilde{\Gamma}$ *has a unique minimal element with respect to inclusion, which is the canonical obstruction* $\Gamma$*.*

Idea of the **proof** from [49]: First, suppose that a first-return map of $\widehat{\mathbb{C}}/\Gamma$ is obstructed, then $g$ has a non-canonical obstruction $\Gamma'$. Suitable annuli of maximal modulus have the property that these moduli are increasing and bounded above. A subsequence of rational maps converges to a limit map on the component in an appropriate normalization. This map has annuli of invariant maximal modulus, so the subdivision into preimages happens parallel to the core curves; this fact is used to obtain type (2, 2, 2, 2). Now obstructions are related to integer eigenvalues of the corresponding matrix lift, and if the degree was not a square, these eigenvalues would be different and have a quotient $> 1$. But then the Thurston matrix of $\Gamma'$ would have $\lambda_{\Gamma'} > 1$ and $\Gamma'$ would be pinching for $g$ as well.

So $\Gamma$ satisfies the assumptions on $\widetilde{\Gamma}$ in Theorem 2.12. Conversely, we must see that any simple obstruction $\widetilde{\Gamma}$ with these properties contains the canonical obstruction $\Gamma$. Since curves of $\Gamma$ and $\widetilde{\Gamma}$ do not intersect according to Theorem 2.11, it remains to show that no periodic component of $\widehat{\mathbb{C}} \setminus \widetilde{\Gamma}$ contains a curve of $\Gamma$. When the first-return map is a Thurston map, this follows from similar arguments as above. When it is a homeomorphism, there would be a Lévy cycle intersecting $\Gamma$ within the component. ■

The assumption that $\widetilde{\Gamma}$ is a simple obstruction is necessary, because otherwise curves from $\widetilde{\Gamma}$ and $\Gamma$ might intersect. Consider the example in Figure 8 of [14], where the spider map of angle 5/12 is mated with its conjugate. Various obstructions are formed by the curves $\alpha$, $\beta$, which surround 2-cycles corresponding to fixed points of $z^2 + \gamma_M(5/12)$, and by $\delta_1$, $\delta_2$, $\delta_3$, $\delta_4$, which surround conjugate postcritical points. Denoting the equator by $\varepsilon$, $\widetilde{\Gamma} = \{\alpha, \beta, \varepsilon\}$ would be a non-simple obstruction with unobstructed component maps, and it does not contain the canonical obstruction $\Gamma = \{\delta_1, \delta_2\}$.

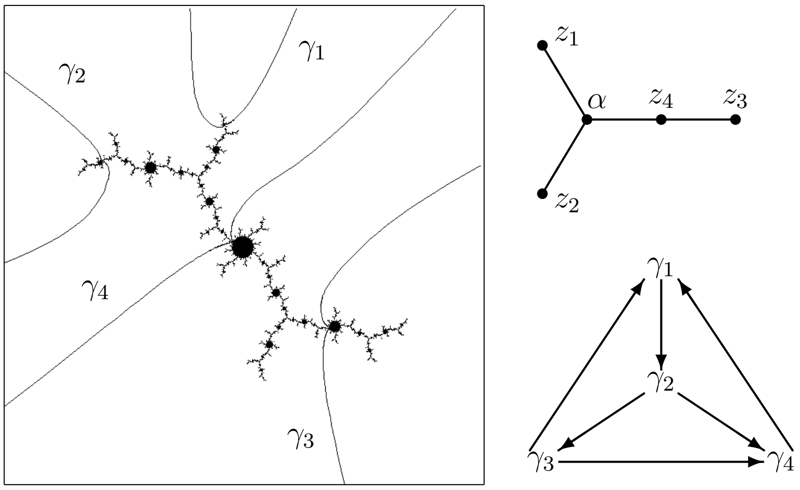
Eigenvalues of non-negative integer matrices appear in two different areas of Thurston's work: the combinatorial characterization of rational maps, and core entropy of quadratic polynomials. $h(c)$ is the topological entropy of $z^2 + c$ on its Hubbard tree $T_c \subset \mathcal{K}_c$. It is computed from the growth factor of iterated preimages, which in the postcritically finite case is the leading eigenvalue $1 \le \lambda \le 2$ of the Markov matrix $A$ describing the mapping of edges under the polynomial. Moreover, it is related to the Hausdorff dimension of biaccessing angles. See [24] and the references therein.

**Proposition 2.13 (Mating conjugate polynomials)**
*Suppose* $p \neq 0$ *is postcritically finite and take the complex conjugate parameter* $q = \overline{p}$*. Consider the canonical obstruction* $\Gamma$ *of the formal mating* $g = P \sqcup Q$*. Then:*

1. *Each* $\gamma \in \Gamma$ *passes through a unique edge of the Hubbard tree* $\varphi_0(T_p)$ *and through the corresponding edge of* $\varphi_\infty(T_q)$ *; there is a unique* $\gamma \in \Gamma$ *for each edge.*

2. *The Markov matrix* $A$ *of* $T_p$ *is the transpose of the Thurston matrix* $M = M_\Gamma$ *of* $g$*, unless different conventions are used in the preperiodic case, with the critical points marked in the Hubbard tree but not in the formal mating: then* $A$ *has an additional eigenvalue of* 0 *compared to* $M$*. The leading eigenvalue* $\lambda$ *is equal in any*

*case, so* $h(p) = \log \lambda_\Gamma$ .




$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

**Figure 3:** The formal mating of the Kokopelli $p$ at $\gamma_M(3/15)$ with its conjugate $q = \overline{p}$, as in Example 4 of [54]. The 4-periodic rays define four loops $\gamma_i$ and a noded surface $\widehat{\mathbb{C}}/\Gamma$ with five pieces. The Thurston matrix $M$ is the transpose of the Markov matrix $A$, which describes the mapping of edges in the Hubbard tree $T_p$ . The leading eigenvalue $\lambda = 1.395337$ with $\lambda^4 - 2\lambda - 1 = 0$ determines the core entropy $h(p) = \log \lambda$.
Under pullback with $g = P \sqcup Q$, the three Lévy cycles converge to ray-equivalence classes: $(\gamma_1\,,\,\gamma_2\,,\,\gamma_3\,,\,\gamma_4)$ gives the original 4-periodic rays, $(\gamma_1\,,\,\gamma_2\,,\,\gamma_3)$ converges to a 3-cycle of loops with 6-periodic rays, and $(\gamma_1\,,\,\gamma_2\,,\,\gamma_4)$ gives the 3-periodic rays from $\alpha_p$ to $\alpha_q$ .

**Proof** of Proposition 2.13: The Hubbard tree $T_p$ is a finite tree with expanding dynamics; critical and postcritical points are marked and there may be additional branch points [8]. Let us assume that the critical points are marked as well in the formal mating $g = P \sqcup Q$. Marking 1 on the 0-ray will not change the canonical obstruction. Now for each edge of $T_p$ choose an arc in the dynamic plane of $P$ passing homotopically transversely through this edge; join it with the complex conjugate arc in the dynamic plane of $Q$ to obtain a simple loop in $\widehat{\mathbb{C}}$ separating marked points of $g$. This works, since the formal mating is constructed by mapping each dynamic plane to a half-sphere. The curves in Figure 3 are dynamic rays in fact; this choice is possible unless $p$ is a direct satellite center, but it is not required here.

1. These loops define a non-empty multicurve $\Gamma$. Under $P$, each edge is covered by either one or two edges, so $\Gamma$ is invariant: in the former case, one of the two preimages of the corresponding loop is inessential. $\Gamma$ is completely invariant in fact, since every edge covers at least one edge. By construction, the Markov matrix $A$ of $T_p$ and the Thurston matrix $M_\Gamma$ agree except for transposition. Now $M_\Gamma$ contains at least one nonzero entry in each row and in each column, so $\Gamma$ is a simple obstruction.

Looking at noded Riemann surfaces in $\mathcal{M}_{G \cdot \Gamma}$ or at the noded topological surface defining the pullback map on $\mathcal{S}_\Gamma$ , the nodes correspond to edges of $T_p$ and the pieces correspond to the vertices of $T_p$ , so they contain a single marked point or branch point of $T_p$ and the corresponding point of $T_q$ . In the former case, there are two marked points of $g$ and at least one node, while in the latter case, there is no marked point of $g$ and at least three nodes.

In the preperiodic case, all first-return maps are homeomorphisms. In the periodic case, the first-return maps for marked points of $T_p$ map a sphere with three or four marked points to itself. In the latter case the two critical points are fixed and another point is fixed as well, while the fourth one goes to that fixed point. This map is unobstructed by the core arc argument: an obstructing curve cannot separate the two critical points, because its preimage would cover it twice, giving an eigenvalue $1/2 < 1$. So an obstruction must surround an arc between the critical points, and its preimages are two curves separating the critical points from one of the other points each, so neither is homotopic to the original curve.

By the Characterization Theorem 2.12, $\Gamma$ contains the canonical obstruction. It remains to show that no proper subset has the same properties. Assuming that $M_\Gamma$ and thus $A$ is reducible in the sense of Perron–Frobenius, it has some block structure understood in terms of simple renormalization according to [24]. For each possible simple renormalization $p = p' * \widehat{p}$, so $p$ belongs to a small Mandelbrot set $p' * \mathcal{M}$ [34], we have a block-triangular structure of $A$ with two diagonal blocks, one corresponding to the periodic parameter $p'$, and an imprimitive one related to the renormalized parameter $\widehat{p}$. Now $A$ maps edges from the big Julia set of $p'$ over edges from the small Julia set of $\widehat{p}$, and $M_\Gamma$ maps small loops to big loops. So taking only the small loops would not give an invariant multicurve, and taking only the big loops would give obstructed component maps: the first-return map corresponds to a mating of conjugate polynomials again. Conversely, although not every possible block decomposition is explained in terms of renormalization, every edge of $T_p$ belongs to a particular level of renormalization, so removing the loop for this edge amounts to removing a block from a particular form with two diagonal blocks.

2. By definition, $\lambda_\Gamma$ is the leading eigenvalue of $M_\Gamma$ and $h(p) = \log \lambda$ with the leading eigenvalue $\lambda$ of $A$. In the preperiodic case, we need not mark the critical points for $g$, and we might also not mark it in $T_p$; then we still have $A$ as the transpose of $M_\Gamma$, since the edge around 0 is mapped two–to–one to the edge before $p$ and the corresponding loop has two homotopic preimages. The matrices will be different using different conventions for $T_p$ and $g$, but according to [24] the leading eigenvalue is the same. ∎

Any multicurve corresponds to a tree in a similar way. A general estimate of $\lambda_\Gamma$ in terms of entropy is given by Shishikura according to [53]. Further applications of trees include the description of Herman rings [51], the construction of maps with Cantor families of circles in the Julia set [17], and the classification of rescaling limits [1]. In recent talks on tropical complex dynamics, Shishikura has suggested to combine arithmetic surgery on a tree with quasiconformal surgery on pieces.

The canonical decomposition of a Thurston map is done in two steps by Bartholdi–Dudko [5]: Lévy cycles generate a decomposition into pieces, such that the first-return maps are homeomorphisms, expanding, or of type (2, 2, 2, 2). The expanding pieces are decomposed again, such that the first-return maps are equivalent to rational ones. Here expansion refers to a suitable metric, which is defined everywhere except at super-attracting cycles. See [6, 18] for various notions of expansion.

## 2.7 The Selinger proof of the Pilgrim Conjecture

For a Thurston map $g$ with canonical obstruction $\Gamma \neq \emptyset$, the first-return maps of $g_\Gamma$ are homeomorphisms or Thurston maps. In [44], Pilgrim has conjectured that the latter are unobstructed when the type is not (2, 2, 2, 2). This was proved by Selinger in Theorem 10.3 of [48] by constructing a subsequence of $(\tau_n)$ with suitable convergence properties. He obtained the more general Characterization Theorem 5.6 of [49] using different techniques; see Theorem 2.12 in Section 2.6. We will need convergence properties from the first proof in Section 3.7. Maybe the following proof shall be read as a complement to the original one in [48], since it focuses on a few points that took me some time to understand, in particular the construction of the uniform bound $D_1$ and dealing with the fact that $\pi_1 : \widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ and $\pi_2 : \widehat{\mathcal{W}} \to \widehat{\mathcal{M}}$ are not covering maps.

The Hurwitz space is defined as $\mathcal{W} = \mathcal{T}/H$, where $H < G$ denotes the subgroup of liftable homeomorphisms: $h \in H$ if there is an $h' \in G$ with $g \circ h' = h \circ g$. Then the cover $\pi : \mathcal{T} \to \mathcal{M}$ factorizes as $\pi = \pi_2 \circ \pi_1$ with $\pi_1 : \mathcal{T} \to \mathcal{W}$ and $\pi_2 : \mathcal{W} \to \mathcal{M}$. Moreover, $H$ has finite index in $G$ and $\pi_2$ is a finite cover. Since $\mathcal{W}$ can be represented by triples of rational maps $f_\tau$ and marked points in its domain and range [14, 22, 31, 32], there is a continuous map $\widetilde{\sigma}_g : \mathcal{W} \to \mathcal{M}$ with $\widetilde{\sigma}_g \circ \pi_1 = \pi \circ \sigma_g$. Now both $G$ and $H$ act by WP-isometries on $\widehat{\mathcal{T}}$, and we have a completion $\widehat{\mathcal{W}} = \widehat{\mathcal{T}}/H$ of Hurwitz space, which is compact. While $\pi$, $\pi_1$, $\pi_2$ are covers and local isometries, the extensions $\pi_1 : \widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ and $\pi_2 : \widehat{\mathcal{W}} \to \widehat{\mathcal{M}}$ are weak contractions; $\pi_1$ is "infinitely branched" and $\pi_2$ is finitely branched. The unique extension $\widetilde{\sigma}_g : \widehat{\mathcal{W}} \to \widehat{\mathcal{M}}$ is Lipschitz continuous with factor $\sqrt{d}$. The strata of $\widehat{\mathcal{W}}$ are labeled by classes $H \cdot \Gamma$ of multicurves. We will not need a concrete description of $\widehat{\mathcal{W}}$ and $\widetilde{\sigma}_g$ by triples.

**Proposition 2.14 (Selinger proof of the Pilgrim Conjecture)**
*Suppose $g$ is a Thurston map of degree $d$ with canonical obstruction $\Gamma \neq \emptyset$, $C$ is a piece of $\widehat{\mathbb{C}}/\Gamma$ mapped to itself by $g_\Gamma$, and the component map $g^C : C \to C$ of degree $\geq 2$ does not have type $(2, 2, 2, 2)$. Fix $\tau_0 \in \mathcal{T}$ and set $\tau_n = \sigma_g^n(\tau_0)$. Then:*

1. *There is sequence $w_i$ in a compact subset of $S_{H\cdot\Gamma} \subset \widehat{\mathcal{W}}$ with $\widetilde{\sigma}_g(w_i) = \pi_2(w_{i+1})$ for $i \geq 0$, and a subsequence $\tau_{n_k}$ with $\pi_1(\sigma_g^i(\tau_{n_k})) \to w_i$ as $k \to \infty$, for all $i \geq 0$.*
2. *Assuming that $\varepsilon(I)$ is sufficiently small and $\varepsilon(I) \searrow 0$ sufficiently fast, there are $k(I)$ and $\widehat{\tau}_I \in \pi_1^{-1}(w_0) \cap S_\Gamma$ with $d_{WP}(\sigma_g^i(\tau_{n_{k(I)}}), \sigma_g^i(\widehat{\tau}_I)) < \varepsilon(I)$ for $0 \leq i \leq I$, such that $\pi_1(\sigma_g^i(\widehat{\tau}_I)) = w_i$ for $0 \leq i \leq I$.*
3. *The component $\sigma_C = \sigma_{g^C}$ of the extended pullback map has a unique fixed point $\widehat{\tau}^C$ and $d_T^C(\sigma_C^i(\widehat{\tau}_I^C), \widehat{\tau}^C) \to 0$ as $i \to \infty$, uniformly in $I \geq i$.*

Intuitively, the situation is as follows: imagine a system of coordinates in a neighborhood of $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$ adapted to a product of three components. The first coordinate becomes 0 as $\Gamma$ is pinched. The second coordinate is related to pieces, where the first-return map is a homeomorphism, and where we do not have convergence. The third coordinate describes pieces where the first-return maps converge. Although such a local product representation of $\widehat{\mathcal{T}}$ is constructed in [42], we do not have any estimates of $\sigma_g$ in that representation. So the proof will be given by constructing various subsequences in an interplay between $\mathcal{T}$, $\widehat{\mathcal{T}}$, and components of $\mathcal{S}_\Gamma$, using both the Teichmüller metric and the Weil–Petersson metric at times. — Note that we have an accumulation statement instead of convergence for two reasons: there may be pieces with at least four marked points and nodes permuted by a homeomorphism, and moreover, $\sigma_g$ is not weakly contracting on $\widehat{\mathcal{T}}$.

**Proof:** 1. By the Canonical Obstruction Theorem 2.11, $\pi(\tau_n)$ accumulates on a compact subset of $\mathcal{S}_{G\cdot\Gamma} \subset \widehat{\mathcal{M}}$; pick an accumulation point $m_0$. Now $\pi_2 : \widehat{\mathcal{W}} \to \widehat{\mathcal{M}}$ is a finite branched cover, so there is a subsequence $(\tau_{n_k^0})_{k\in\mathbb{N}_0} \subset (\tau_n)_{n\in\mathbb{N}_0}$ and a $w_0 \in \pi_2^{-1}(m_0)$ with $\pi_1(\tau_{n_k^0}) \to w_0$ as $k \to \infty$; we have $w_0 \in \mathcal{S}_{H\cdot\Gamma} \subset \widehat{\mathcal{W}}$ since precisely the curves in $\Gamma$ are pinched as $n_k^0 \to \infty$. Then $\pi(\sigma_g(\tau_{n_k^0})) \to m_1 = \widetilde{\sigma}_g(w_0)$ by continuity. Choose a subsequence $\tau_{n_k^1}$ of $\tau_{n_k^0}$ with $\pi_1(\sigma_g(\tau_{n_k^1})) \to w_1 \in \pi_2^{-1}(m_1)$. Define $m_i$, $w_i$, and subsequences $\tau_{n_k^i}$ inductively, then any subsequence $\tau_{n_k}$ of the diagonal sequence $\tau_{n_k^k}$ satisfies the claim.

To obtain the bound $D_1$ below, this subsequence is constructed as follows: For $\tau$ on a smooth curve from $\tau_0$ to $\tau_1$, there is a lower bound $l(\gamma,\, \sigma_g^n(\tau)) \ge R$, $\gamma \notin \Gamma$. Take constants $C_*$ for $\mathcal{T} = \mathcal{T}_g$ according to item 5 of Proposition 2.2 and $D_* = D_*(R)$ for $\mathcal{T}^C = \mathcal{T}_{g^C}$ according to item 6. Pick intermediate points $\tau_0^0 = \tau_0$, $\tau_0^1, \dots, \tau_0^U = \tau_1$ on the curve with $d_T(\tau_0^{u-1},\, \tau_0^u) \le D_*/C_*$ for $1 \le u \le U$. Now choose the indices $n_k$ for the subsequence $\tau_{n_k}$ of $\tau_{n_k^k}$ such that there are limits $\pi_1(\sigma_g^{n_k}(\tau_0^u)) \to w_0^u$ as $k \to \infty$.

2. Intuitively, if $\mathcal{N}$ is a small neighborhood of $w_0$ in $\widehat{\mathcal{W}}$, we have $\pi_1(\tau_{n_k}) \in \mathcal{N}$ for sufficiently large $k$, and we shall find $\widehat{\tau}_0 \in \pi_1^{-1}(w_0)$ close to $\tau_{n_k}$ for some $k$. Then the curves of $\widetilde{\Gamma} \in G \cdot \Gamma$ are short in a neighborhood of $\widehat{\tau}_0$ and only the curves in $\Gamma$ are short at $\tau_{n_k}$, so $\widetilde{\Gamma} = \Gamma$ and $\widehat{\tau}_0 \in \mathcal{S}_\Gamma$. A subgroup $G_\Gamma < G$ acts transitively on the fiber $\pi_1^{-1}(w_0) \cap \mathcal{S}_\Gamma$; it leaves $\Gamma$ invariant and it is generated by Dehn twists about curves in pieces of $\widehat{\mathbb{C}}/\Gamma$ and about curves of $\Gamma$. The latter act trivially on the stratum but not trivially in a neighborhood, thus $\pi_1$ is not a cover. See also Remark 2.9. In a neighborhood of $\widehat{\tau}_0$, $\pi_1$ maps to the quotient with respect to $H \cap G_\Gamma$, $\widehat{\mathcal{T}}$ is a product of disks and of half-disks plus the center point, and $\pi_1$ is infinite–to–one locally on the half-disks at those points. So $\pi_1$ is not a local WP-isometry at $\widehat{\tau}_0$: we have $d_{WP}(\pi_1(\tau'),\, \pi_1(\tau)) = \min d_{WP}(h \cdot \tau',\, \tau) \le d_{WP}(\tau',\, \tau)$, and arbitrarily close to $\widehat{\tau}_0$ there are $\tau', \tau \in \mathcal{T}$ with $d_{WP}(\pi_1(\tau'),\, \pi_1(\tau)) < d_{WP}(\tau',\, \tau)$. But $\widehat{\mathcal{T}}$ is a unique geodesic space and the geodesic from $\tau_{n_k}$ to $\widehat{\tau}_0$ is contained in $\mathcal{T}$ except for the endpoint [58]. All $h \cdot \tau_{n_k}$ close to $\widehat{\tau}_0$ are related by global isometries fixing $\widehat{\tau}_0$, thus

$$d_{WP}(\tau_{n_k},\, \widehat{\tau}_0) = d_{WP}(\pi_1(\tau_{n_k}),\, w_0) = d_{WP}(\pi(\tau_{n_k}),\, m_0) \; . \tag{3}$$

So we may define $\mathcal{N}$ in terms of a small distance $\varepsilon(0)$ to $w_0$ and have the same radius in components of $\pi_1^{-1}(\mathcal{N})$. Before stating the actual construction of $\widehat{\tau}_0$, note that we want to have a shadowing property for a finite number $I$ of steps; this is possible since $\sigma_g$ is Lipschitz continuous, and both $k(I)$ and $\widehat{\tau}_I$ will depend on $I$.

Assume that $\varepsilon(I) \searrow 0$ for $0 \le I \to \infty$ and $(\sqrt{d}+1)\varepsilon(I)$ is less than the minimal distance in the fiber $\pi_2^{-1}(m_i) \cap \mathcal{S}_{H\cdot\Gamma}$ for $0 \le i \le I$. Moreover, the preimage of an $\varepsilon(I)$-neighborhood of $w_i$ under $\pi_1$ shall have disjoint components, where $\pi_1$ is described explicitly as an infinite–to–one map in terms of $H \cap G_\Gamma$ as explained above. Then for $I \ge 0$ there are $k(I)$ and $\widehat{\tau}_I \in \pi_1^{-1}(w_0) \cap S_\Gamma$ with $d_{WP}(\sigma_g^i(\tau_{n_{k(I)}}),\, \sigma_g^i(\widehat{\tau}_I)) < \varepsilon(I)$ for $0 \le i \le I$. A finite induction shows $\pi_1(\sigma_g^i(\widehat{\tau}_I)) = w_i$ for $0 \le i \le I$: assuming the claim for $i-1$, we have $\pi(\sigma_g^i(\widehat{\tau}_I)) = \widetilde{\sigma}_g(\pi_1(\sigma_g^{i-1}(\widehat{\tau}_I))) = \widetilde{\sigma}_g(w_{i-1}) = m_i$ and

$$\begin{aligned} & d_{WP}(w_i,\, \pi_1(\sigma_g^i(\widehat{\tau}_I))) \\ \le\;& d_{WP}(w_i,\, \pi_1(\sigma_g^i(\tau_{n_k}))) + d_{WP}(\pi_1(\sigma_g^i(\tau_{n_k})),\, \pi_1(\sigma_g^i(\widehat{\tau}_I))) \end{aligned}$$

$$
\begin{aligned}
&\le\ d_{WP}(m_i\,,\,\pi(\sigma_g^i(\tau_{n_k}))) + d_{WP}(\sigma_g^i(\tau_{n_k})\,,\,\sigma_g^i(\widehat{\tau}_I)) \\
&\le\ d_{WP}(\widetilde{\sigma}_g(\pi_1(\sigma_g^{i-1}(\widehat{\tau}_I)))\,,\,\widetilde{\sigma}_g(\pi_1(\sigma_g^{i-1}(\tau_{n_k})))) + d_{WP}(\sigma_g^i(\tau_{n_k})\,,\,\sigma_g^i(\widehat{\tau}_I)) \\
&\le\ (\sqrt{d}+1)\,\varepsilon(I)\ .
\end{aligned} \tag{4}
$$

Concerning the first term in (4), we have discarded $\pi_2^{-1}$, which is not a weak contraction in general. But in this case $d_{WP}(w_i\,,\,\pi_1(\sigma_g^i(\tau_{n_k}))) = d_{WP}(m_i\,,\,\pi(\sigma_g^i(\tau_{n_k})))$ by the same arguments as for (3). Finally, by the assumption on the distance in the fiber $\pi_2^{-1}(m_i)$, the claim is proved for $i$.

3. All maps, sets, and elements related to the stratum $\mathcal{S}_\Gamma \in \widehat{\mathcal{T}}$ have a product structure describing pieces of the noded Riemann surfaces; the component for the piece corresponding to $C$ is denoted by a superscript or subscript $C$. Since the length of hyperbolic geodesics is continuous on $\widehat{\mathcal{T}}$, we have $l(\gamma,\,\sigma_C^i(\widehat{\tau}_I^C)) \ge R$ for $0 \le i \le I$ and all essential simple closed curves $\gamma$ in $C$. The same lower bound holds for simple closed geodesics in the corresponding piece of the noded surface defined by $\pi_2(w_0^u)$. Now $\sigma_g$ is weakly contracting on $\mathcal{T}$ with respect to $d_T$, so the intermediate points satisfy $d_T(\sigma_g^{n_k}(\tau_0^{u-1}),\,\sigma_g^{n_k}(\tau_o^u)) \le D_*/C_*$ and $d_{WP}(\sigma_g^{n_k}(\tau_0^{u-1}),\,\sigma_g^{n_k}(\tau_o^u)) \le D_*$. For $I \ge I_*$ there are $\widehat{\tau}_I^u \in \pi_1^{-1}(w_0^u) \cap S_\Gamma$ with $d_{WP}(\sigma_g^{n_{k(I)}}(\tau_0^u),\,\widehat{\tau}_I^u) < \varepsilon(I) \le D_*/2$. Then $d_{WP}(\widehat{\tau}_I^{u-1},\,\widehat{\tau}_I^u) < 2D_*$ and the same estimate holds for the components related to $C$. Since these components are intermediate points between $\widehat{\tau}_I^C$ and $\sigma_C(\widehat{\tau}_I^C)$ with geodesic length $\ge R$, item 6 of Proposition 2.2 gives

$$
d_T(\widehat{\tau}_I^C,\,\sigma_C(\widehat{\tau}_I^C)) < 2U \cdot 1/4 = U/2 \le D_1\ . \tag{5}
$$

Here $D_1 \ge U/2$ is chosen such that the estimate holds not only for $I \ge I_*$ but for all $I \ge 1$. Now suppose $0 \le i < I$. The pullback map $\sigma_C$ is weakly contracting, so $\sigma_C^i(\widehat{\tau}_I^C)$ and $\sigma_C^{i+1}(\widehat{\tau}_I^C)$ are connected with an arc of T-length $\le D_1$, on which the length of simple closed geodesics is bounded below by $R \cdot \mathrm{e}^{-D_1}$. This condition defines compact subsets of $\mathcal{M}_C$ and $\mathcal{W}_C$ according to the Mumford Proposition 2.2.7. The contraction of $\sigma_C$ at $\tau^C$ depends only on $f_{\tau^C}$ or $\pi_1^C(\tau^C)$, so some iterate of $\sigma_C$ is uniformly strictly contracting over the compact subset of $\mathcal{W}_C$, since $g^C$ is not of type (2, 2, 2, 2). For notational convenience, let us assume that the first iterate suffices. So there is $0 < L < 1$ with

$$
d_T(\sigma_C^i(\widehat{\tau}_I^C),\,\sigma_C^{i+1}(\widehat{\tau}_I^C)) \le D_1 \cdot L^i \qquad \text{and} \qquad d_T(w_i^C\,,\,w_{i+1}^C) \le D_1 \cdot L^i \tag{6}
$$

for $0 \le i < I$; the second estimate is true for all $i \ge 0$ since $w_i$ is independent of $I > i$. By completeness of $\mathcal{W}^C$ we have limits $w_i^C \to \widehat{w}^C$ and $m_i^C \to \widehat{m}^C$ with $\widetilde{\sigma}_C(\widehat{w}^C) = \pi_2^C(\widehat{w}^C) = \widehat{m}^C$ and an estimate

$$
d_T(w_i^C\,,\,\widehat{w}^C) \le \frac{D_1}{1-L}\,L^i\ . \tag{7}
$$

Take a lower bound $\widehat{\varepsilon}$ for the T-distance in the fiber $(\pi_1^C)^{-1}(\widehat{w}^C)$, and such that an $\widehat{\varepsilon}/3$-neighborhood of $\widehat{w}^C$ has disjoint preimages under the cover $\pi_1^C$. Choose $i = i_*$ such that the right hand side of (7) is less than $\widehat{\varepsilon}/3$. Pick $I > i$ and find $\widehat{\tau}^C \in (\pi_1^C)^{-1}(\widehat{w}^C)$ with $d_T(\sigma_C^i(\widehat{\tau}_I^C),\,\widehat{\tau}^C) < \varepsilon/3$. Then

$$
d_T(\widehat{\tau}^C,\,\sigma_C(\widehat{\tau}^C)) \le d_T(\widehat{\tau}^C,\,\sigma_C^i(\widehat{\tau}_I^C)) + d_T(\sigma_C^i(\widehat{\tau}_I^C),\,\sigma_C^{i+1}(\widehat{\tau}_I^C)) + d_T(\sigma_C^{i+1}(\widehat{\tau}_I^C),\,\sigma_C(\widehat{\tau}^C))\ ,
$$

which is bounded by $\widehat{\varepsilon}/3 + \widehat{\varepsilon}/3 + \widehat{\varepsilon}/3$. So $\widehat{\tau}^C$ is a fixed point of $\sigma_C$. Since $g^C$ is not of type (2, 2, 2, 2), it is unobstructed, the fixed point is unique, and with $D = D_1/(1-L)$ we have $d_T(\sigma_C^i(\widehat{\tau}_I^C),\, \widehat{\tau}^C) \le D \cdot L^i$ for all $i_* \le i < I < \infty$. Actually, an estimate of this form is valid as well, when only some iterate of $\sigma_C$ is strictly contracting. ∎

# 3 Convergence with collisions

The main result of the present paper describes a fairly general situation, where the canonical obstruction $\Gamma$ of a bicritical Thurston map $g$ is identified, and colliding marked points are shown to converge.

## 3.1 Essential equivalence and convergence properties

When $\Gamma$ is a multicurve such that most components of $\widehat{\mathbb{C}} \setminus \Gamma$ are disks, can we assume that $g$ maps disks to disks?

**Lemma 3.1 (Preimage of a multicurve)**
*Suppose $g$ is a bicritical Thurston map of degree $d \ge 2$ with marked critical points, $\Gamma$ is a completely invariant multicurve, and there is a distinguished component $C$ of $\widehat{\mathbb{C}} \setminus \Gamma$, such that all components $\widetilde{C} \ne C$ are disks. Then the following conditions are equivalent:*

- *the preimage $g^{-1}(C)$ is connected;*
- *for each component $\widetilde{C} \ne C$, $g^{-1}(\widetilde{C})$ is a union of disks; and*
- *each disk $\widetilde{C} \ne C$ contains at most one of the two critical values of $g$.*

*Under these conditions, the essential preimages of curves in $\Gamma$ form a multicurve $\Gamma'$, and there is a homeomorphism $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ isotopic to the identity with $\varphi(\Gamma') = \Gamma$.*

The **proof** is straightforward by noting that each curve $\gamma \in \Gamma$ surrounds an arc connecting marked points, and by considering an arc beween the critical values. ∎

If $\Gamma$ contains at least two curves, this implies already that each disk component $\widetilde{C} \ne C$ of $\widehat{\mathbb{C}} \setminus \Gamma$ is mapped to a disk by $g \circ \varphi$; under the conditions below this also holds in the case of one curve, where $C$ is a disk itself. So by collapsing all $\gamma \in \Gamma$, $g \circ \varphi$ becomes a continuos self-map of the noded topological surface $\widehat{\mathbb{C}}/\Gamma$. As a map between the pieces corresponding to disks $\widetilde{C} \ne C$, this defines a homeomorphism or a bicritical $d$-to-1 cover in each case. The definition of the **component map** for $C$ is more involved, since $C$ contains additional disks bounded by inessential curves in $g^{-1}(\Gamma)$, which are mapped to disks $\widetilde{C} \ne C$ by $g \circ \varphi$. So take any continuous $\varphi_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ collapsing each $\widetilde{C} \ne C$ to a point and choose $\varphi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that it collapses each disk bounded by an inessential preimage of $\gamma \in \Gamma$ to a point, while $\varphi_1 = \varphi_0$ except in disjoint neighborhoods of these closed disks. Then there is a unique $g^C$ with $\varphi_0 \circ (g \circ \varphi) = g^C \circ \varphi_1$. When one of the latter disks contains a marked point $z$, we must assume $\varphi_1(z) = \varphi_0(z)$ in addition, so that $g^C$ is postcritically finite.

**Proposition 3.2 (Essential equivalence)**
*Consider a bicritical Thurston map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \ge 2$, with marked set $Z$ including the critical points. Suppose there is a completely invariant multicurve*

*$\Gamma \neq \emptyset$ and a component $C$ of $\widehat{\mathbb{C}} \setminus \Gamma$ such that:*

• *All components $\widetilde{C} \neq C$ of $\widehat{\mathbb{C}} \setminus \Gamma$ are disks; these disks are preperiodic or periodic under $g$ after an isotopic deformation $\varphi$, and the periodic disks are mapped homeomorphically.*

• *$C$ is mapped to itself with degree $d$ when disks bounded by inessential preimages of $\Gamma$ are collapsed; the component map $g^C$ constructed above shall be combinatorially equivalent to a rational map $f$.*

*Then $\Gamma$ is the canonical obstruction of $g$, which is determined uniquely up to homotopy by $g$ alone.*

*We shall say that $C$ is the* ***essential component*** *of $\widehat{\mathbb{C}} \setminus \Gamma$, $\widetilde{g} = g^C$ is the* ***essential map****, and $g$ is* ***essentially equivalent*** *to $f$.*

**Proof:** According to the discussion of Theorem 2.10, $C$ is considered as a piece of a noded surface and is related to a component of the invariant stratum $\mathcal{S}_{G\cdot\Gamma} \subset \widehat{\mathcal{M}}$, and $\widetilde{g} = g^C$ is a component map of $g_\Gamma$. All disks correspond to pieces, which are attached directly to $C$. If one of these pieces contains a critical point, the node will be critical as well, and the piece is mapped with degree $d$; by assumption it is preperiodic. Accordingly, a periodic critical point must belong to $C$. Both critical points are allowed to be in pieces corresponding to disks $\widetilde{C} \neq C$, but not in the same piece, because then $\widetilde{g}$ would have degree one.

Now $\Gamma$ consists of one or several Lévy cycles and all of their essential preimages, so it is a simple obstruction. The component map $g^C = \widetilde{g}$ is unobstructed, since it is combinatorially equivalent to the rational map $f$, which is not a flexible Lattès map. The remaining first-return maps are homeomorphisms, so according to the Characterization Theorem 2.12, $\Gamma$ contains the canonical obstruction. Assuming that the canonical obstruction was smaller, we would enlarge $C$ by omitting one or several Lévy cycles from $\Gamma$, and $g^C$ would be obstructed. So $\Gamma$ is the canonical obstruction of $g$. ■

Considering the Thurston Algorithm $\tau_n = \sigma_g^n(\tau_0)$ with rational maps $f_n$ sending point configurations at time $n+1$ to those at time $n$, we know that curves corresponding to $\Gamma$ will be pinched and points will collide; by Theorem 2.11, $(\tau_n) = (\sigma_g^n(\tau_0))$ accumulates at most on $\mathcal{S}_\Gamma \subset \widehat{\mathcal{T}}$ and $\pi(\tau_n)$ accumulates on a compact subset of $\mathcal{S}_{G\cdot\Gamma}$. Normalizing three points to $\infty$, 0, 1, a component of $\widehat{\mathbb{C}} \setminus \Gamma$ is singled out, and all other components will shrink to points. We shall see that in the present situation, when the normalization singles out the essential component $C$, marked points do not wander, but they have limits as $n \to \infty$. So there is a limiting point configuration, such that not all points are distinct, and a rational map $f$ with $f_n \to f$. The following Theorem 3.3 has applications to quadratic matings, anti-matings [29], encaptures [26], and spiders. Its proof is based on the extension of $\sigma_g$ to augmented Teichmüller space $\widehat{\mathcal{T}}$, but I have tried to formulate the assumptions in terms of components of $\widehat{\mathbb{C}} \setminus \Gamma$ instead of pieces of $\widehat{\mathbb{C}}/\Gamma$, so that it may be applied in other papers without introducing $\widehat{\mathcal{T}}$.

**Theorem 3.3 (Convergence of marked points and rational maps)**
*Consider a bicritical Thurston map $g$, a multicurve $\Gamma \neq \emptyset$ and a component $C$ of $\widehat{\mathbb{C}} \setminus \Gamma$, such that $g$ is essentially equivalent to a rational map $f$ according to Proposition 3.7. Use a normalization of critical points at 0 and $\infty$, and another*

*marked point at* $1$*, which is arbitrary in* $C$ *or in a disk not containing a critical point. Normalize* $f$ *analogously. Then the Thurston Algorithm* $\sigma_g$ *for the unmodified map* $g$ *with any initial* $\tau_0 \in \mathcal{T}$ *satisfies:*

*If* $f$ *is not of type* $(2, 2, 2, 2)$*, we have* $f_n \to f$*. The marked points converge to preperiodic and periodic points of* $f$*; two points collide if and only if they belong to the same disk* $\widetilde{C} \neq C$ *in* $\widehat{\mathbb{C}} \setminus \Gamma$*.*

*Analogous statements hold for a path* $\tau_t$ *with* $\tau_{t+1} = \sigma_g(\tau_t)$*.*

A typical example is provided by a formal mating $g = P \sqcup Q$ of quadratic polynomials, having ray connections between postcritical points but no cyclic ray connections: then $\Gamma$ consists of curves around postcritical ray-equivalence classes. See the example in Figure 2 right. These are the only obstructions according to [54], so the essential mating $\widetilde{g}$ is unobstructed and combinatorially equivalent to a rational map $f$, excluding type $(2, 2, 2, 2)$ for now. Then Theorem 3.3 gives $f_n \to f$ for the pullback defined by the unmodified formal mating; see Section 4.3 for details. According to [27], the convergence statement is wrong in general when $\widetilde{g}$ and $f$ are of type $(2, 2, 2, 2)$.

**Remark 3.4 (Essential equivalence and convergence)**
*The notion of essential equivalence shall emphasize that* $g$ *itself determines the canonical obstruction* $\Gamma$*, the essential map* $\widetilde{g}$*, and the rational map* $f$*, and no modification is needed to ensure* $f_n \to f$*.*

*For simplicity I have considered the bicritical case only, because* $f_n$ *is determined explicitly by its critical values; a more general result is conceivable, where the canonical obstruction* $\Gamma$ *is given and all first-return maps of periodic components are homeomorphisms or bicritical. An application would be given by renormalized matings between conjugate limbs, which is a joint research project with Arnaud Chéritat. E.g., in Figure 2 left, the first-return maps are equivalent to* $z^2-2$ *and the Thurston Algorithm* $\sigma_g$ *for the formal mating* $g$ *is expected to satisfy* $f_n(f_{n+1}(z)) \to z^2 - 2$*.*

*To obtain a more general result with multicritical component maps, it should also be checked whether collisions between critical points are allowed; examples of topological matings with this kind of collisions are given by Meyer [37].*

## 3.2 Local convergence in configuration space

Intuitively, what happens is that point configurations $x_i(n)$ cluster according to the disks containing $z_i$, and the pullback of clusters determined by $\sigma_g$ should stay close to the pullback of single points obtained from $\sigma_f$. This argument involves interchanging limits, and I have not been able to prove it with direct estimates. So, convergence will be proved in Section 3.7 in two steps: first use augmented Teichmüller space and the Selinger Proposition 2.14 for a global result, accumulation at the prospective limit $x^\infty$. Then a local result is applied, attraction according to Proposition 3.7 below. The global result is needed to get close to $x^\infty$ and to distinguish different fixed points of the extended pullback relation. And the local result is needed because the global one provides accumulation only, not convergence, as explained in Section 2.7. Since the general proof of local attraction requires some lengthy machinery, let us first look at two examples of quadratic matings, assuming a few definitions from Section 4.2.

branch portrait

**Example 3.5 (with collisions apart from critical points)**
*For the formal mating $P \sqcup Q$ with $P(z) = z^2 + \mathrm{i}$ and $Q(z) = z^2 - 1$ according to Figure 1, this reads*

$$x_1' = \pm\sqrt{\frac{x_1 - x_2}{1 - x_2}} \quad x_2' = \pm\sqrt{\frac{2x_1}{1 + x_1}} \; . \tag{8}$$

**Example 3.6 (including collisions at critical points)**
*Figure 2 right*

The pullback of point configurations may be visualized as a kind of movie: as time $t$ or $n$ flows, the points $x_i(n)$ move within a single copy of $\widehat{\mathbb{C}}$. To formulate neighborhoods, convergence, or derivatives of point configurations, it is more convenient to consider $x = (x_1\,, \dots, x_{|Z|})$ as a single point in $\widehat{\mathbb{C}}^{|Z|}$.

**Proposition 3.7 (Local attraction)**
*Consider $g$, $\Gamma$, and $f$ according to Theorem 3.3. Assume in addition that the marked point normalized to 1 is chosen more restrictively: if a marked point is identified with a critical point, then 1 represents a critical value, whose iterates are not identified with a critical point. Using the notations introduced above we have:*

1. *If no marked point is identified with a critical point, a branch of the pullback relation (9) extends analytically to a neighborhood of $x^\infty$ in $\mathbb{C}^{|Z|-3}$. The eigenvalues $\lambda$ of the derivative at the fixed point $x^\infty$ are of the following form: they are eigenvalues of $D\sigma_f$ at $[\mathrm{id}]$, or $\lambda = 0$, or $\lambda^{rp} = \rho^{-r}$ when an $rp$-cycle of $g$ in a $p$-cycle of disks corresponds to a $p$-cycle of $f$ having multiplier $\rho$.*
2. *If $f$ is not of type $(2,\, 2,\, 2,\, 2)$, there is a neighborhood $\mathcal{N}$ of $x^\infty$ in $\widehat{\mathbb{C}}^{|Z|-3}$, which is attracting in the following sense: when $\tau_t$ is a path in $\mathcal{T}$ with $\tau_{t+1} = \sigma_g(\tau_t)$, and $\pi_3(\pi(\tau_t)) \in \mathcal{N}$ for a segment of $t$-length 1, then the path in configuration space will stay in $\mathcal{N} \cap \pi_3(\mathcal{M})$ forever and converge to $x^\infty \in \mathcal{N} \setminus \pi_3(\mathcal{M})$.*

The **proof** is given in Sections 3.3–3.6. Note that when $\Gamma$ would be replaced with a non-homotopic multicurve $\widehat{\Gamma} \in G \cdot \Gamma$, which is grouping marked points in the same way, the new $\tilde{g}$ may be obstructed or equivalent to a different $f$. On the other hand, we will not use information on the homotopy class of $\Gamma$ in the proof of Proposition 3.7. Here the key point is the assumption on the path in item 2: for a different $\widehat{\Gamma}$, the attracting neighborhood $\mathcal{N}$ would be the same, but there may be no path segment of length 1 contained in it. The same remark applies to the usual Thurston Algorithm without identifications in fact: the pullback relation will have several attracting fixed points, and one of these is chosen depending on the isotopy class of $g$ or on the initial path segment.

## 3.3 Local extension without collisions at critical points

**Notations and basic properties** of the pullback relation: The branch portrait of $g$ defines a map $\#$ of indices, such that marked points of $g$ are mapped as $g(z_i) = z_{i\#}$. The Thurston Algorithm provides sequences of homeomorphisms $\psi_n$ and rational maps $f_n$ with $f_n \circ \psi_{n+1} = \psi_n \circ g$. The point configuration $x(n)$ at time $n$ has the components $x_i(n) = \psi_n(z_i)$; sometimes these are called marked points as well. The basic relation is $f_n(x_i(n+1)) = x_{i\#}(n)$.

The coordinates depend on the normalization, and we have specific indices $\alpha'$, $\beta'$, $\gamma'$ with $x_{\alpha'}(n) = \infty$, $x_{\beta'}(n) = 0$, and $x_{\gamma'}(n) = 1$ for all $n$. Assume that $z_{\alpha'}$ and $z_{\beta'}$ are the critical points of $g$. The normalization fixes an embedding $\pi_3 : \mathcal{M} \to \widehat{\mathbb{C}}^{|Z|}$; its range consists of $(x_1, \ldots, x_{|Z|})$ with pairwise distinct components, and $\infty$, 0, 1 at specific positions, which is open and dense in a subset $\mathbb{C}^{|Z|-3} \subset \widehat{\mathbb{C}}^{|Z|}$.

Denote $\alpha = \alpha'\#$, $\beta = \beta'\#$, $\gamma = \gamma'\#$, then $f_n$ is determined by $f_n(\infty) = x_\alpha(n)$, $f_n(0) = x_\beta(n)$, and $f_n(1) = x_\gamma(n)$ as a Möbius transformation of $z^d$. This gives the pullback relation in the form of a multi-valued function from $\pi_3(\mathcal{M})$ into itself:

$$f_n^{-1}(z) = \sqrt[d]{\frac{x_\gamma(n) - x_\alpha(n)}{x_\gamma(n) - x_\beta(n)} \cdot \frac{z - x_\beta(n)}{z - x_\alpha(n)}} \qquad x_i' = \sqrt[d]{\frac{x_\gamma - x_\alpha}{x_\gamma - x_\beta} \cdot \frac{x_{i\#} - x_\beta}{x_{i\#} - x_\alpha}} \tag{9}$$

The second formula may be considered either as a multi-valued function $x \mapsto x'$, or as a step of the pullback $x(n) \mapsto x(n+1)$. By the normalization we have $|Z| - 3$ independent variables in the domain and $|Z| - 3$ variables in the range; when some $x_m = \infty$, the corresponding factors cancel from the radicand. For each value of the index $i$, the radicand is either constant $\infty$, 0, 1 or never $\infty$, 0, 1. In the Thurston pullback, a specific branch of the $d$-th root is chosen by the isotopy class of $\psi_{n+1}$ or by continuity of the path $x_i(t)$. Note that unless all marked points $z_i$ of $g$ are periodic, we have pairs of indices $i \neq k$ with $g(z_i) = g(z_k)$, so $i\# = k\#$. Then $x_i(n+1)$ and $x_k(n+1)$ are given by different branches of a root with the same radicand, and the number of variables could be reduced by replacing $x_k$ with $\zeta \cdot x_i$ for some $\zeta$ with $\zeta^d = 1$ determined by $g$; while this makes sense for a concrete example, it would complicate the notation for the present discussion.

**Relating $g$ to $f$:** To describe the Thurston pullback $\sigma_f$, denote the marked points of $f$ by $\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_{|\widetilde{Z}|})$ and define indices with $\widetilde{z}_{\widetilde{\alpha}'} = \infty$, $\widetilde{z}_{\widetilde{\beta}'} = 0$, $\widetilde{z}_{\widetilde{\gamma}'} = 1$, and $\widetilde{z}_{\widetilde{\alpha}} = f(\infty)$, $\widetilde{z}_{\widetilde{\beta}} = f(0)$, $\widetilde{z}_{\widetilde{\gamma}} = f(1)$. Each marked point of $f$ corresponds either to a unique marked point of $g$ in $C$ or to a unique disk component $\widetilde{C} \neq C$ of $\widehat{\mathbb{C}} \setminus \Gamma$ with at least two marked points; this defines a surjection $D : \{1, \ldots, |Z|\} \to \{1, \ldots, |\widetilde{Z}|\}$. The normalizations are assumed to be compatible, i.e., $D(\alpha') = \widetilde{\alpha}'$, $D(\beta') = \widetilde{\beta}'$, and $D(\gamma') = \widetilde{\gamma}'$.

Denote a specific diagonal of $\widehat{\mathbb{C}}^{|Z|}$ by $\Delta_\Gamma$: it contains all $x = (x_1, \ldots, x_{|Z|})$ with $x_i = x_k$ if and only if $D(i) = D(k)$, and $x_{\alpha'} = \infty$, $x_{\beta'} = 0$, $x_{\gamma'} = 1$. The extension of $\pi_3$ to $\widehat{\mathcal{M}}$ satisfies $\pi_3(\mathcal{S}_{G \cdot \Gamma}) = \Delta_\Gamma$; this is an isomorphism when all disks have only two marked points, but it is forgetful otherwise. Note that there is a natural bijection between $\Delta_\Gamma \subset \widehat{\mathbb{C}}^{|Z|}$ and the configuration space of $\sigma_f$ contained in $\widehat{\mathbb{C}}^{|\widetilde{Z}|}$: the components of $x$ have repetitions, such that $x$ corresponds to $\widetilde{x}$ with $x_i = \widetilde{x}_{D(i)}$ for all $i$. In particular, the point configuration $x^\infty$ with repetitions given by $x_i^\infty = \widetilde{z}_{D(i)}$ is the prospective limit of $x(n)$.

The Thurston pullback $\sigma_f$ defines a multi-valued pullback relation on its configuration space, which is given by formulas analogous to (9). A specific branch of this pullback relation has a fixed point at $\widetilde{x} = \widetilde{z}$; it is analytic in a neighborhood of $\widetilde{z}$ in $\mathbb{C}^{|\widetilde{Z}|-3} \subset \widehat{\mathbb{C}}^{|\widetilde{Z}|}$. A simple but suggestive observation is the following: under the bijection of $x$ and $\widetilde{x}$ the conjugate pullback map in a neighborhood of $x^\infty$ in $\Delta_\Gamma$ is given by choosing a suitable branch in (9). The reason is that when $z_i$ is in the disk corresponding to $\tilde{z}_j$, then $g(z_i)$ is in the disk corresponding to $f(\tilde{z}_j)$. And if $z_i$ and $z_k$ are in the same disk, their images are both in one disk. So for $x \in \Delta_\Gamma$ the radicands with different indices $i\#$ and $k\#$ agree whenever $D(i) = D(k)$. Note that this does not mean that the local branch extends to a neighborhood of $x^\infty$ in $\widehat{\mathbb{C}}^{|Z|-3}$: this is the case precisely when the radicand cannot become 0 or $\infty$. If, e.g., there is an index $k \neq \beta'$ with $D(k) = D(\beta') = \widetilde{\beta}'$, so $x_k$ will be identified with 0, the radicand may be 0 within any neighborhood of $x^\infty$ in $\widehat{\mathbb{C}}^{|Z|-3}$. (On $\Delta_\Gamma$, the radicand in (9) is constant 0 both for $i = k$ and $i = \beta'$, which is not a problem.)

The additional assumption restricts the choice of the index $\gamma'$ with $x_{\gamma'}(n) = \psi_n(z_{\gamma'}) = 1$, when a critical point $\omega$ of $g$ is identified with another marked point, i.e., they are in the same disk component $\widetilde{C} \neq C$ of $\widehat{\mathbb{C}} \setminus \Gamma$. Then $\omega$ is strictly preperiodic and all forward iterates of $\omega$ will belong to disks as well; preimages of $\omega$ may be marked or not, and in the former case, may undergo identifications or not. In general we may take the critical value $g(\omega)$ for $z_{\gamma'}$, unless it equals the other critical point or is identified with it or with a preimage. In that case we can take the other critical value, which is not identified with a preimage of $\omega$, since the critical points of $f$ are not periodic.

## 3.4 Local proof without collisions at critical points

**Proof of Proposition 3.7:** Now after choosing $\gamma'$ and $\widetilde{\gamma}' = D(\gamma')$, which determines the term of $f$, we shall consider three cases of increasing complexity:

Case 1: no marked point is identified with a critical point.

Case 2: a marked point is identified with a critical point, but no postcritical point is identified with a critical point.

Case 3: for some $k \geq 1$, $g^k(z_{\alpha'})$ is identified with $z_{\beta'}$. This means that $f^k(\infty) = 0$, but $g$ has disjoint critical orbits.

When $g^k(z_{\beta'})$ is identified with $z_{\alpha'}$ instead, this does not require separate arguments, since $f$ and $f_n$ are related to case 3 by a conjugation with $z \mapsto 1/z$.

**Case 1:** Recall that $\tilde{z}$ denotes the marked points $\tilde{z}_j$ of $f$ and $x^\infty \in \Delta_\Gamma$ with $x_i^\infty = \tilde{z}_{D(i)}$ is our prospective limit of $x(n)$. Since the marked points of $f$ are preimages of marked points under pullback with suitable branches of $f^{-1}$, for each $i \neq \alpha'$, $\beta'$ there is a unique branch in (9), such that taking components of $x = x^\infty$ for the radicand, gives $x_i' = x_i^\infty$ for the root. These branches extend analytically to $x$ in a neighborhood of $x^\infty$ in $\mathbb{C}^{|Z|-3}$, which means $|x_i - x_i^\infty| < \varepsilon$ for $i = 1, \ldots, |Z|$ with $i \neq \alpha'$, $\beta'$, $\gamma'$, since the radicands are varying in small neighborhoods of values distinct from 0 and $\infty$.

To determine the eigenvalues of the derivative matrix for this branch of the pullback relation, we shall obtain block matrices by choosing new coordinates labeled $u = (u_1, \ldots, u_{|\widetilde{Z}|})$ and $v = (v_{|\widetilde{Z}|+1}, \ldots, v_{|Z|})$ as follows:

- For each $j = 1, \ldots, |\widetilde{Z}|$, choose one index $i$ with $D(i) = j$ and set $x_i = u_j$. If there are $k \neq i$ with $D(k) = j$, set $x_k = u_j + v_m$ for an unused index $m$. When all variables are defined successively, note that $x = x^\infty$ corresponds to $u = \tilde{z}$ and $v = 0$.
- For $j = \widetilde{\gamma}'$, the marked point at 1 shall be $u_j$, i.e., $x_{\gamma'} = u_{\widetilde{\gamma}'}$. The corresponding choice for 0 and $\infty$ is satisfied here anyway, because these are not identified with other marked points, but it is required explicitly in cases 2 and 3.

• Renumber $v_{|\widetilde{Z}|+1}\,, \ldots, v_{|Z|}$ such that preperiodic marked points $u_j + v_m$ appear before periodic ones, higher preperiods before lower preperiods, and the periodic marked points are grouped according to their cycles, with a natural order within each cycle. We may renumber the components of $x$ such that $x_j = u_j$ or $x_i = u_{D(i)} + v_i$ , respectively.

Since three components of $u$ are constant, the local branch $(u,\, v) \mapsto (u',\, v')$ of the pullback relation still has $(|\widetilde{Z}| - 3) + (|Z| - |\widetilde{Z}|) = |Z| - 3$ free variables. We shall see that the derivative at the fixed point $(\tilde{z},\, 0)$ has the block-triangular form

$$D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \qquad \text{with} \qquad B = \begin{pmatrix} R & Q \\ 0 & P \end{pmatrix} \; . \tag{10}$$

Note that the local manifold with $u \approx \tilde{z}$ and $v = 0$ is invariant under the pullback relation: $u'_j$ and $u'_j + v'_i$ correspond to marked points of $g$ in the same disk, so their images belong to one disk as well. If $v = 0$, the radicands determining $u'_j$ and $u'_j + v'_i$ agree, and the local branches of the roots agree, so $v'_i = 0$. This argument shows that the lower left block of $D$ is 0, and the upper left block $A$ coincides with the derivative of the local pullback relation for $\sigma_f$ in configuration space, which is analytically conjugate to $\sigma_f$ in a neighborhood of its fixed point in Teichmüller space. So $A$ has attracting eigenvalues $\lambda$ unless $f$ is of type (2, 2, 2, 2) — then one eigenvalue will be neutral.

Before discussing the block structure of $B$, let us consider partial derivatives of the pullback relation (9) more explicitly. A few components of $(u,\, v)$ determine the bicritical rational map $f_{uv}$ by its critical values and the image of 1. For suitable combinations of indices we have

$$f_{uv}(u'_j + v'_i) = u_l + v_k \qquad \text{and} \qquad f_{uv}(u'_j) = u_l + v_m \tag{11}$$

in general; either $v_k$ or $v_m$ may be missing. Total differentials at $(u,\, v) = (\tilde{z},\, 0)$ read

$$[\ldots] + f'(\tilde{z}_j)\cdot(\mathrm{d}u'_j + \mathrm{d}v'_i) = \mathrm{d}u_l + \mathrm{d}v_k \qquad \text{and} \qquad [\ldots] + f'(\tilde{z}_j)\cdot\mathrm{d}u'_j = \mathrm{d}u_l + \mathrm{d}v_m \;, \tag{12}$$

where $[\ldots]$ denotes differentials involving the partial derivatives of $f_{uv}(z)$ with respect to the parameters $(u,\, v)$. Observing that these expressions agree for both equations when setting $u = u' = \tilde{z}$ and $v = v' = 0$, the difference gives

$$f'(\tilde{z}_j) \cdot \mathrm{d}v'_i = \mathrm{d}v_k - \mathrm{d}v_m \; . \tag{13}$$

Note again that either $v_k$ or $-v_m$ may be missing. The argument remains valid, if a component $v_i$ , $v_k$ , $v_m$ appears in the parameters of $f_{uv}$ as well, or if $u'_j$ or $u_l$ is 1. Now (13) shows again that the lower left block of $D$ is 0. The blocks $R$ and $P$ of $B$ in (10) refer to preperiodic and periodic marked points, respectively. In (13), $v'_i$ has a higher preperiod than $v_k$ and $v_m$ , or these refer to a cycle of disks. Thus $R$ is strictly upper triangular, with 0 on the diagonal, and its eigenvalues $\lambda$ vanish.

Finally, consider the blocks of $P$, which are related to periodic cycles of $g$. Since periodic disks of $\widehat{\mathbb{C}} \setminus \Gamma$ are mapped homeomorphically by $g \circ \varphi$, each disk in a cycle has the same number of marked points, which are permuted by the first-return map. The obvious examples are two cycles of periodic points with the same period identified pairwise, or a single cycle of high period identified such that a cycle of lower period results for $f$. All possibilities are covered by the following description: a cycle of $f$ has period $p \geq 1$ and there are one or several cycles of $g$ with periods $rp$ for possibly different values $r \geq 1$. Consider two scenarios:

First, $g$ has a $p$-cycle in the disks under consideration, which is labeled $u_1, \ldots, u_p$. The indices are chosen to illustrate the order in the $v$-blocks; they do not actually start with 1. Every other $rp$-cycle of $g$, $r \geq 1$, is described by $v_i$ in the following order: $u_1+v_1 \ldots u_p+v_p$, $u_1+v_{p+1} \ldots u_p+v_{2p}$, $\ldots$, $u_1+v_{(r-1)p+1} \ldots u_p+v_{rp}$. Then $P$ contains an $rp$-block on the diagonal with nonzero entries $1/f'(\tilde{z}_j)$ only directly above the diagonal and in the lower left corner, since $\pm v_m$ is absent from (11)–(13). By rescaling all variables, this block becomes a companion matrix and the entry in the lower left position is $(1/f'(\tilde{z}_1) \cdot \ldots \cdot 1/f'(\tilde{z}_p))^r = \rho^{-r}$. So the characteristic polynomial is $\lambda^{rp} - \rho^{-r}$. Note that $|\rho| > 1$ because $f$ is postcritically finite and all periodic points are superattracting or repelling, so $|\lambda| < 1$.

Second, all cycles of $g$ within the $p$-cycle of disks have periods $rp$ with $r > 1$, then the $u_j$ must belong to one of these cycles; choose this cycle to be last among the cycles of $g$ in the current cycle of components. Starting with index 1 again for simplicity, it is labeled as $u_1 \ldots u_p$, $u_1 + v_1 \ldots u_p + v_p$, $\ldots$, $u_1 + v_{(r-2)p+1} \ldots u_p + v_{(r-1)p}$. Now $f_{uv}(u'_p) = u_1 + v_1$ shows that $\pm v_m$ is no longer absent from (11)–(13). We have a block of size $(r-1)p$ on the diagonal of $P$, with nonzero entries $1/f'(\tilde{z}_j)$ directly above the diagonal, and further entries $-1/f'(\tilde{z}_p)$ in rows $p$, $2p$, $\ldots$, $(r-1)p$ of the first column. After rescaling all variables appropriately, this is the companion matrix of $\lambda^{(r-1)p} + \rho^{-1}\lambda^{(r-2)p} + \ldots + \rho^{-(r-1)} = \frac{\lambda^{rp}-\rho^{-r}}{\lambda^p-\rho^{-1}}$. Note that $A$ contains a cyclic $p$-block for $u_1, \ldots, u_p$ again, but since $A$ is not block-triangular, it need not have eigenvalues with $\lambda^p = \rho^{-1}$. If $g$ has further $r'p$-cycles in the same components of $\widehat{\mathbb{C}} \setminus \Gamma$, these are treated according to the first scenario, giving $\lambda^{r'p} = \rho^{-r'}$; there will be additional entries above the diagonal blocks, which do not contribute to the characteristic polynomial of $P$, but may prevent $P$ from being diagonalizable. — An alternative approach to the second scenario would be to modify $g$ isotopically so that it has a $p$-cycle in the disks, and to mark this cycle in addition.

So if $f$ is not of type (2, 2, 2, 2), all eigenvalues of the extended pullback relation $x \mapsto x'$ at the fixed point $x^\infty$ are attracting. We shall construct a norm on $\mathbb{C}^{|Z|-3}$, such that the linearization satisfies $\|x' - x^\infty\| \leq L\|x - x^\infty\|$ for some $L < 1$; in a small neighborhood $\mathcal{N}$ with $\|x - x^\infty\| < \delta$ the pullback map is analytic and satisfies $\|x' - x^\infty\| \leq L'\|x - x^\infty\|$ for some $L < L' < 1$. To define this norm, conjugate the derivative matrix to its Jordan normal form by a linear change of variables. Rescale components such that the entries 1 above the diagonal become $\varepsilon$, and choose $\varepsilon > 0$ small such that the new matrix is contracting with respect to the standard Euclidean norm. The norm $\|\cdot\|$ in the original coordinates corresponds to this Euclidean norm.

## 3.5 Local proof including collisions at critical points

## 3.6 Local proof including collisions at critical points

In the remaining cases 2 and 3, we will not have an analytic branch of $x \mapsto x'$ in a neighborhood of $x^\infty$, but we shall construct an attracting neighborhood for the path $x(t)$ nevertheless when $f$ is not of type (2, 2, 2, 2). In **case 2**, suppose that $z_m$ is identified with a critical point $\omega$. If $g^k(z_{\alpha'}) = z_{\beta'}$ and $\omega = z_{\alpha'}$, then $g^k(z_m)$ is identified with $z_{\beta'}$, and we shall redefine $\omega = z_{\beta'}$ and $m$ such that $z_m$ is identified with $\omega$. Preimages of $z_m$ and $\omega$ may be marked or not, and identified or not. Define new coordinates $(u,\, v,\, w)$ with $w$ representing all $x_i$, such that $z_i$ is identified with a critical or precritical point, and $u$, $v$ describing the remaining $x_i$ as in case 1. The marked point $x_{\gamma'}$ normalized to 1 may be precritical but not be identified with a precritical or critical point. So the rational maps $f_{uv}$ will not depend on $x_m$ and its preimages; the multi-valued pullback relation $(u,\, v,\, w) \mapsto (u',\, v',\, w')$ is such that $u'$ and $v'$ do not depend on $w$. As in case 1, we have a local analytic branch and an attracting neighborhood $\mathcal{N}_0$ for $(u,\, v) \mapsto (u',\, v')$.

Now the pullback for $x_m$ is asymptotic to $x'_m \sim \sqrt[d]{c \cdot v_j}$ or $x'_m \sim \sqrt[d]{c/v_j}$ when $\omega = z_{\beta'}$ or $\omega = z_{\alpha'}$, respectively. The branch of the root is defined uniquely along a path, but there is no analytic branch in a neighborhood of 0 or $\infty$. Moreover, this expression does not seem to be attracting, but it is used for a preperiodic point here. So we only need it to be continuous in the sense that $v_j \to 0$ implies $\sqrt[d]{c \cdot v_j} \to 0$ or $\sqrt[d]{c/v_j} \to \infty$ for any branch. For $(u,\, v) \in \mathcal{N}_0$, $x'_m$ will be in a small neighborhood of 0 or $\infty$, and its preimages will be in small neighborhoods of precritical points of $f$. The product of $\mathcal{N}_0$ with these neighborhoods defines the attracting neighborhood $\mathcal{N}$ for $(u,\, v,\, w)$. Note that the coordinates $u$ and $v$ converge geometrically as $\mathcal{O}(L^t)$, and $w$ with $\mathcal{O}(L^{t/d})$, or $\mathcal{O}(L^{t/d^2})$ if $f^k(\infty) = 0$ or $f^k(0) = \infty$.

In **case 3**, $f^k(\infty) = 0$ for some $k \ge 1$, and the postcritical point $z_m = g^k(z_{\alpha'})$ is identified with $z_{\beta'}$. Consequently, iterates of $z_m$ are identified with corresponding iterates of $z_{\beta'}$. If preimages of $z_{\beta'}$ are identified with preimages of $z_m$ as well, or if additional non-postcritical marked points are identified with critical or precritical points, they are labeled $w$ and treated separately as in case 2 — these points will be ignored from now on. The pullback relation is not reducible in case 3: we have $x'_m \sim \sqrt[d]{c \cdot v_j}$ again, and this coordinate cannot be treated separately, since it is pulled back to the critical value $x_\alpha$. This value appears in the parameters of $f_{uv}$ and influences the pullback of every point. Postcritical variables $v$ may appear directly in these parameters, if $k = 1$ or $f(0)$ has preperiod 1. Note that $x'_m \sim \sqrt[d]{c \cdot v_j}$ is the only component of (9) not analytic in a neighborhood of $x = x^\infty$ or $(u,\, v) = (\tilde{z},\, 0)$.

Choose the coordinates $(u,\, v)$ such that preperiodic iterates of $z_{\beta'}$ are of type $u$ and preperiodic iterates of $z_m$ are of type $u + v$, including $x_m = v_m$. So $v'_m \sim \sqrt[d]{c \cdot v_j}$ and the partial derivative $\partial v'_m / \partial v_j \to \infty$ as a branch of the root is continued analytically along the path. In a way, the matrix $D$ in (10) has a unique infinite entry, in block $R$ and above the diagonal. One idea to deal with this is to use the orbifold metric of $f$ for $x_i \approx x_i^\infty = \tilde{z}_{D(i)}$ instead of the usual metric on $\mathbb{C}$. Alternatively, we may lift the path and the pullback relation to new coordinates $(U,\, V)$ with $U_i = u_i$ for all $i$ and, e.g., $V_m = v_m$ but $V_j^d = v_j$; this gives $V'_m \sim \sqrt[d]{c} \cdot V_j$, where the branch of $\sqrt[d]{c}$ is determined by the chosen lift of a concrete path. The $v$-coordinates of iterates must be lifted as well, but this gives an analytic pullback relation only when $g$ has a postcritical cycle of the same period $p$ as $f$. If we are in the second scenario, simply add a $p$-cycle to $g$ within the cycle of disks; this does not change the pullback of the other points in the $x$-coordinates, but when the new points are used as $u$-coordinates, this allows to estimate the $v$-coordinates. — In the lifted coordinates, we have attracting eigenvalues and an attracting neighborhood as in case 1.

In any case, the neighborhood $\mathcal{N}$ can be chosen such that its images under different branches of the pullback relation are either contained in $\mathcal{N}$ or disjoint from it. Since path segments are appended continuously, the given segment stays in $\mathcal{N}$ forever and is attracted. ■

### 3.7 Convergence of the Thurston Algorithm

Completing the **proof of Theorem 3.3:** The marked point $z_{\gamma'}$ normalized to $x_{\gamma'} = 1$ is chosen such that together with the critical points at $x_{\beta'} = 0$ and $x_{\alpha'} = \infty$, it singles out the component $C$. The associated embedding $\pi_3 : \mathcal{M} \to \widehat{\mathbb{C}}^{|Z|}$ extends continuously to $\pi_3 : \widehat{\mathcal{M}} \to \widehat{\mathbb{C}}^{|Z|}$ according to Proposition 2.8.3; on $\mathcal{S}_{G.\Gamma}$ it is described as follows: Each $m \in \mathcal{S}_{G.\Gamma}$ defines a noded Riemann surface; the piece corresponding to $C$ is isomorphic to $\widehat{\mathbb{C}}$. The isomorphism is unique by sending specific marked points and nodes to 0, 1, $\infty$. Now marked points in other pieces are sent to the same points as the corresponding nodes. So the fixed point $\widehat{\tau}^C$ of $\sigma_C = \sigma_f$ has the following property: all $\tau \in \mathcal{S}_\Gamma$ with component $\tau^C = \widehat{\tau}^C$ have $\pi_3(\pi(\tau)) = x^\infty$.

Now suppose that the choice of $x_{\gamma'} = 1$ was made according to the restrictions from Proposition 3.7, and obtain an attracting neighborhood $\mathcal{N}$. Combine spherical metrics to define a metric $d$ on $\widehat{\mathbb{C}}^{|Z|-3} \subset \widehat{\mathbb{C}}^{|Z|}$ and choose $\delta > 0$ such that the open ball of radius $2\delta$ around $x^\infty$ is contained in $\mathcal{N}$. Proposition 3.7 applies with the same notation of $g$, $\Gamma$, $C$, and $\widehat{\tau}^C$. We shall start by constructing a path in $\mathcal{T} \cup \mathcal{S}_\Gamma$; for suitable $0 < i < I < \infty$ it goes from $\tau_{n_{k(I)}+i}$ to $\sigma_g^i(\widehat{\tau}_I)$, to $\sigma_g^{i+1}(\widehat{\tau}_I)$, and to $\tau_{n_{k(I)}+i+1}$ .

There is a T-ball around $\widehat{\tau}^C$ in $\mathcal{T}^C$, such that all $\tau \in \mathcal{S}_\Gamma$ with $\tau^C$ in this ball satisfy $d(\pi_3(\pi(\tau)),\, x^\infty) < \delta$. Choose $i$ according to Proposition 2.14.3 such that $\sigma_C^i(\widehat{\tau}_I^C)$ and $\sigma_C^{i+1}(\widehat{\tau}_I^C)$ belong to this ball for $I > i$. Since $\widehat{\mathcal{M}}$ is compact, $\pi_3$ is uniformly continuous. Choose $I > i$ such that $\varepsilon(I)$ is sufficiently small, so $d_{WP}(m',\, m) < \varepsilon(I)$ implies $d(\pi_3(m'),\, \pi_3(m)) < \delta$. Now we have $d_{WP}(\tau_{n_{k(I)}+i}\, ,\, \sigma_g^i(\widehat{\tau}_I)) < \varepsilon(I)$ and $d_{WP}(\sigma_g^{i+1}(\widehat{\tau}_I),\, \tau_{n_{k(I)}+i+1}) < \varepsilon(I)$ according to Proposition 2.14.2. The first and third segments of our preliminary path shall be the corresponding WP-geodesics. The middle segment from $\sigma_g^i(\widehat{\tau}_I)$ to $\sigma_g^{i+1}(\widehat{\tau}_I)$ shall be the product of T-geodesics in the components of $\mathcal{S}_\Gamma$.

So with $n_* = n_{k(I)}+i$ we have constructed a preliminary path from $\tau_{n_*}$ to $\tau_{n_*+1}$ in $\mathcal{T} \cup \mathcal{S}_\Gamma$, such that $d(\pi_3(\pi(\tau)),\, x^\infty) < 2\delta$ on this path. Since the ball is open and the path is compact, we may choose a nearby path from $\tau_{n_*}$ to $\tau_{n_*+1}$ in $\mathcal{T} \cap (\pi_3 \circ \pi)^{-1}(\mathcal{N})$. The pullback of this path interpolates $(\tau_n)_{n \ge n_*}$ and projects to a path in $\pi_3(\mathcal{M})$, which stays in $\mathcal{N}$ and converges to $x^\infty$ according to Proposition 3.7.

In the course of these proofs, several paths were constructed and discarded to obtain convergence of the sequence $\pi_3(\pi(\tau_n))$. Now suppose that a path $\tau_t$ is given from the start. Then for $\varepsilon > 0$ we want to find $T \ge 0$ with $d(\pi_3(\pi(\tau_t)),\, x^\infty) < \varepsilon$ for $t > T$. This is done by applying the result for sequences to the pullback of finitely many intermediate points on the initial segment, which are chosen depending on $\varepsilon$, such that each smaller segment gives a change $< \varepsilon/2$ in $\pi_3(\mathcal{M})$. Note again that the T-distance and thus the WP-distance stays bounded under the pullback and that $\pi_3$ is uniformly continuous on $\widehat{\mathcal{M}}$.

Finally, consider the case where the marked point normalized to 1 does not satisfy the assumption of Proposition 3.7. Then we have convergence in a different normalization, and the two normalizations are related by a rescaling with a convergent factor. ■

**Remark 3.8 (Rate of convergence)**
*1. The attracting eigenvalues at $x^\infty$ in configuration space were related to multipliers of $f$ and to eigenvalues of $D\sigma_f$ in Proposition 3.7, where $\sigma_f$ includes both postcritical points and additional marked points. Note that similar estimates apply to collisions with critical points, and that additional marked points of $g$ without collisions converge to marked points of $f$ with a rate determined by multipliers of $f$ as well. The orbifold metric [41, 34] provides uniform expansion and uniform estimates for multipliers of $f$, especially $|\rho| \ge k^p$ for the multiplier of a non-postcritical $p$-cycle. So if $L_f$ is a bound for the eigenvalues of $D\sigma_f$ without additional marked points, then $d(\pi_3(\pi(\tau_n)),\, x^\infty)$ asymptotically shrinks exponentially by $\max(k^{-1},\, L_f) < 1$ independently of the number of additional marked points with or without collisions.*

*2. This bound on eigenvalues does not directly imply uniform convergence, e.g., in the case of a formal mating $g$ with fixed $\psi_0$ but an arbitrary number of marked points: using a standard distance on $\widehat{\mathbb{C}}^{|Z|-3}$, the norm of the derivative may be arbitrarily large when eigenvectors are nearly parallel. Moreover, the number of initial steps to get into a neighborhood of $x^\infty$ may grow with $|Z|$. See [11] for results on uniform convergence.*

*3. Under the assumptions of essential equivalence according to Proposition 3.7 and Theorem 3.3, the leading eigenvalue is always $\lambda_\Gamma = 1$. In a different situation with $\lambda_\Gamma > 1$, collisions shall happen faster than exponentially.*

# 4 Construction and convergence of mating

We shall employ five different notions of mating: the formal mating is constructed explicitly, modified to an essential mating, and it is combinatorially equivalent to a rational map, the combinatorial mating. This is a geometric mating at the same time, since it is conjugate to the topological mating, which is defined as a quotient of the formal mating or of the polynomials in turn. While the notion of the geometric mating may be most natural, the construction best understood starts with the formal and combinatorial matings in the postcritically finite case. — Convergence properties of the formal mating are discussed in Section 4.3 in a direct application of Theorem 3.3.

## 4.1 Dynamics and combinatorics of quadratic polynomials

The dynamics of a quadratic polynomial $f_c(z) = z^2 + c$ is understood as follows: all $z$ with large modulus escape to $\infty$ under the iteration; the non-escaping points form the filled Julia set $\mathcal{K}_c$. By definition, the parameter $c$ belongs to the Mandelbrot set $\mathcal{M}$, if $\mathcal{K}_c$ is connected, or equivalently, if the critical orbit does not escape. Then the Boettcher map $\Phi_c : \widehat{\mathbb{C}} \setminus \mathcal{K}_c \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ maps dynamic rays $\mathcal{R}_c(\theta)$ to straight rays with angle $\theta$ [41, 47, 39].

When $\theta$ is periodic or preperiodic under doubling, the landing point $z = \gamma_c(\theta) \in \partial\mathcal{K}_c$ is periodic or preperiodic under $f_c$ as well. In the parameter plane, parameters $c = \gamma_M(\theta) \in \partial\mathcal{M}$ are defined as landing points of parameter rays with rational angles. If $\theta$ is periodic, $c$ is the root of a unique hyperbolic component with a unique center; for that parameter, the critical orbit is periodic. Preperiodic angles give Misiurewicz parameters, for which the critical value is preperiodic. Dynamic rays landing together are important for ray connections. For parameters $c$ in a limb of $\mathcal{M}$, the fixed point $\alpha_c$ has unique angles and a unique rotation number, while the other fixed point always satisfies $\beta_c = \gamma_c(0)$.

## 4.2 Definitions and construction of matings

The **formal mating** $g = P \sqcup Q$ of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ is a Thurston map, which is conjugate to $P$ and $Q$ on the lower and upper half-spheres, respectively. E.g., consider an odd map $\varphi_0 : \mathbb{C} \to \mathbb{D}$ with $\varphi_0(r \cdot \mathrm{e}^{\mathrm{i}\theta}) \to \mathrm{e}^{\mathrm{i}\theta}$ as $r \to \infty$ and set $\varphi_\infty(z) = 1/\varphi_0(z)$; then define $g = \varphi_0 \circ \ P \circ \varphi_0^{-1} \cup \varphi_\infty \circ Q \circ \varphi_\infty^{-1}$. A simple explicit choice is given by $\varphi_0(z) = z/\sqrt{|z|^2 + 1}$, then $g$ will be smooth but not quasi-regular. **External rays** of $g$ are unions of $\varphi_0(\mathcal{R}_p(\theta))$ and $\varphi_\infty(\mathcal{R}_q(-\theta))$ together with a point on the equator; **ray-equivalence classes** are maximal connected unions of rays and landing points in $\varphi_0(\partial\mathcal{K}_p)$ and $\varphi_\infty(\partial\mathcal{K}_q)$. Their geometry is described in [25]. According to [54, 52], in the postcritically finite case there are:

- Cyclic ray connections corresponding to non-removable Lévy cycles, when the parameters $p$ and $q$ are in conjugate limbs of the Mandelbrot set.
- Otherwise only trees giving identifications within and between Julia sets, maybe in several steps.
- If postcritical or additional marked points are in the same ray-equivalence class, these are surrounded by removable Lévy cyles. Then an **essential mating** $\widetilde{g}$ is defined by modifying $g$: these trees or disks are collapsed to points and the map is modified at preimages as well, giving an unobstructed Thurston map with a smaller number of marked points [54, 52].

The Thurston algorithm for $g$ gives a sequence of homeomorphisms $\psi_n$ and of rational maps $f_n$ with $\psi_n \circ g = f_n \circ \psi_{n+1}$. The homeomorphisms $\psi_n$ converge up to isotopy, unless $g$ is obstructed or of type (2, 2, 2, 2). The following result is classical:

**Theorem 4.1 (Combinatorial mating by Rees–Shishikura–Tan)**
*Suppose the polynomials $P$ and $Q$ are postcritically finite and the parameters are not in conjugate limbs of the Mandelbrot set. Then the formal mating $g = P \sqcup Q$ does not have a non-removable obstruction, and the **combinatorial mating** $f$ is obtained as follows:*

*a) If the formal mating $g$ does not have a removable obstruction, then $f$ is combinatorially equivalent to $g$, and the Thurston Algorithm for $g$ converges $f_n \to f$.*

*b) If the formal mating $g$ has a removable obstruction, then $f$ is defined as the rational map equivalent to the essential mating $\widetilde{g}$. The Thurston Algorithm for $\widetilde{g}$ converges $f_n \to f$, unless $\widetilde{g}$ is of type* (2, 2, 2, 2).

We may speak of "the" combinatorial mating, since Möbius conjugacy classes are avoided by assuming a normalization: the critical point 0 of $f$ corresponds to $P$, the critical point $\infty$ to $Q$, and 1 is the fixed point of argument 0. Different combinatorial matings might still be conjugate to each other by marking a different fixed point, or by interchanging $P$ and $Q$. E.g., the combinatorial matings of $z^2 \pm \mathrm{i}$

with the Basilica $z^2 - 1$ are distinct, but conjugate by a rotation of the fixed points. This ambiguity is avoided with the alternative normalization $f(\infty) = 1$.

**Idea of the proof:** In [54] it is shown that every obstruction of a quadratic Thurston map contains a removable Lévy cycle, or there is a "good" Lévy cycle. The curves are homotopic to periodic ray-equivalence classes. See the example in Figure 3. Removable cycles correspond to loops around simply connected ray-equivalence classes, while cyclic ray connections indicate the presence of non-removable Lévy cycles. Then there is a good Lévy cycle corresponding to closed ray connections between the two $\alpha$-fixed points, which exist precisely when the parameters are in conjugate limbs. Otherwise the essential mating $\widetilde{g}$ can be defined as a branched cover, which is unobstructed.

When $\widetilde{g}$ has type not (2, 2, 2, 2), the proof of existence and uniqueness is completed with the Thurston Theorem 2.5, but the case of type (2, 2, 2, 2) is different: here the absence of obstructions for $\widetilde{g}$ is not sufficient to guarantee that there is an equivalent rational map $f$. The criterion requires a real-affine lift of $\widetilde{g}$ instead [14, 22, 34, 50, 6]. Here the proof can be given by applying the Shishikura Algorithm to each essential mating in question, to determine the matrix of the lift and to check that the eigenvalues are not real. The cases of $1/4 \sqcup 1/4$ and $1/6 \sqcup 1/6$ are described in [54], five more cases are discussed in [40], and the remaining two cases are settled in [27]. ■

For degree $d \geq 3$ analogous definitions are used, but there is no combinatorial characterization of cyclic ray connections in general. Obstructions need not contain Lévy cycles, and non-removable obstructions may exist also when there are no cyclic ray connections [53]. Moreover, the combinatorial mating will not be unique in the case of flexible Lattès maps.

The **topological mating** $P \coprod Q$ is defined by collapsing all rational and irrational ray-equivalence classes to points. Alternatively, take the disjoint union of $\mathcal{K}_p$ and $\mathcal{K}_q$ and consider the equivalence relation generated by $\gamma_p(\theta) \sim \gamma_q(-\theta)$. By the Moore Theorem [43], we have a Hausdorff space homeomorphic to the sphere, when the ray-equivalence relation of $P \sqcup Q$ is closed and not separating. Then the formal mating $g = P \sqcup Q$ descends to a branched cover of the quotient space, so the topological mating $P \coprod Q$ is a branched cover of the sphere, which is defined up to conjugation. When mating polynomials from conjugate limbs of $\mathcal{M}$, the topological mating does not exist because the sphere would be pinched. It may happen that there is not even a Hausdorff space; examples of unbounded closed ray-equivalence classes are given in [25].

Now the **geometric mating** is a rational map $f$ topologically conjugate to the topological mating, $f \cong P \coprod Q$. In the postcritically finite case of mating, the following result from [52, 12] shows that every combinatorial mating from non-conjugate limbs is a geometric mating in fact. Moreover, the topological mating exists and there are no cyclic irrational ray connections either. Note that in contrast to Theorems 4.1 and 4.3, type (2, 2, 2, 2) does not require special considerations.

**Theorem 4.2 (Rees–Shishikura)**
*For postcritically finite quadratic polynomials $P$ and $Q$, with $p$ and $q$ not in conjugate limbs of $\mathcal{M}$, consider the formal mating $g = P \sqcup Q$ and the essential mating $\widetilde{g}$. According to Theorem 4.1, the combinatorial mating $f$ is combinatorially equivalent to $\widetilde{g}$. Moreover:*

1. *There is a semi-conjugation $\Psi_\infty$ from $g$ to $f$.*

2. *$\Psi_\infty$ maps ray-equivalence classes to points, and it is injective otherwise. So the topological mating is defined on a Hausdorff space homeomorphic to the sphere, and $\Psi_\infty$ descends to a topological conjugation from the topological mating $P \coprod Q$ to $f$. Now the geometric mating of $P$ and $Q$ exists and it is given by $f$.*

3. *$\Psi_\infty$ is a uniform limit of homeomorphisms.*

Let us look at details from the proof for later reference in Section 4.3. Note that the Thurston Theorem 2.5 and its application in Theorem 4.1 used convergence of isotopy classes $[\psi_n]$ for an arbitrary $\psi_0$, but now we need convergence of maps $\Psi_n$ for a special choice of $\Psi_0$.

Idea of the **proof:** 1. The simplest case concerns preperiodic $P$ and $Q$ without postcritical identifications. There are isotopic $\Psi_0$, $\Psi_1$ with $\Psi_0 \circ g = f \circ \Psi_1$ and a path $\Psi_t \in [\Psi_0]$ with $\Psi_t \circ g = f \circ \Psi_{t+1}$ for $0 \le t < \infty$. Since $f$ is uniformly expanding with respect to the orbifold metric [41, 34], the homotopic length of a segment $\{\Psi_t(z) \,|\, n \le t \le n+1\}$ shrinks exponentially in $n$, uniformly in $z$. So $\Psi_\infty = \lim \Psi_n$ is continuous, surjective, and a semi-conjugation.

The second case includes hyperbolic $P$ or $Q$. Then the orbifold metric is more singular at periodic critical orbits, and exponential shrinking is uniform away from these only. Although $\Psi_0$ can be chosen as a local conjugation at these cycles by employing the Böttcher conjugation, this does not guarantee $\Psi_1 = \Psi_0$ there. In [52] the latter property is obtained by modifying $\Psi_0$ with suitable Dehn twists. In [12], $\Psi_0$ is left unchanged but its pullback is described locally in terms of Dehn twists.

The third case requires the construction of an essential mating $\widetilde{g}$ by collapsing a family $Y$ of critical and postcritical ray-equivalence classes. Then we have $\Psi_t = \widetilde{\Psi}_t \circ \pi_t$, where $\widetilde{\Psi}_t$ is a pseudo-isotopy for the essential mating. For $n \le t < n+1$, $\pi_t$ collapses the ray-equivalence classes in $g^{-n}(Y)$ to points independent of $t$. So restricted to $n \le t \le n+1$, $\Psi_t$ is a pseudo-isotopy outside of finitely many ray-trees.

2. The homotopic length of subarcs of rays shrinks exponentially, at least away from precritical and postcritical classes. Moreover, any ray-equivalence class has a finite number of rays, so its image under $\Psi_t$ shrinks to a point. It is quite involved to show that distinct classes are mapped to distinct points by $\Psi_\infty$ [52].

3. When $\widetilde{g} = g$, $\Psi_\infty$ is the end of the pseudo-isotopy $\Psi_t$. Otherwise both $\widetilde{\Psi}_n$ converges to a continuous map and $\Psi_n$ converges to $\Psi_\infty$, but $\Psi_n = \widetilde{\Psi}_n \circ \pi_n$ is not a homeomorphism. So the projections $\pi_n$ are approximated by homeomorphisms, observing that ray-equivalence classes are collapsed successively. ■

Now $\gamma_f(\theta) = \Psi_\infty \circ \varphi_0 \circ \gamma_p(\theta) = \Psi_\infty \circ \varphi_\infty \circ \gamma_q(-\theta)$ is a semi-conjugation from the angle doubling map on $\mathbb{R}/\mathbb{Z}$ to $f$ on its Julia set, which is not injective but can be approximated by embeddings; it gives a Peano curve when both $P$ and $Q$ are preperiodic. Then $\gamma_f$ maps the Brolin measure on $\mathbb{R}/\mathbb{Z}$ to the Lyubich measure on $\widehat{\mathbb{C}}$. A tiling is obtained from $T = \gamma_f([0,\, 1/2])$ as well: then the Julia set is $T \cup (-T)$, while $T \cap (-T)$ is the image of the spines [40]. In general this gives no finite subdivision rule. Alternative constructions with a pseudo-equator [36] or Hubbard trees [56] are possible in certain cases.

## 4.3 Convergence properties of the formal mating

According to the discussion of Theorem 3.3, there is no need to correct a removable obstruction by identifying marked points manually: it will be removed automatically during the iteration of the unmodified Thurston Algorithm, in the sense that several marked points have the same limit, at least in the non-(2, 2, 2, 2) case. Then $[\psi_n]$ diverges in Teichmüller space, but the images of marked points and the rational maps $f_n$ converge. Now the Thurston Algorithm can be implemented for the formal mating without dealing with the combinatorics and topology of postcritical ray-equivalence classes; the essential mating is used only as a step in the proof, but not in the actual pullback. See [11] and Section 5 for a discussion of slow mating. Actually, the same technique gives identifications for all repelling periodic and preperiodic points by marking them in addition. As conjectured in [7, 10], e.g., the proof is based on the Selinger extension to augmented Teichmüller space in Section 2.5.

**Theorem 4.3 (Convergence of maps & rational ray-equivalence classes)**
*Consider the Thurston Algorithm $[\psi_n]$ with any initial $\psi_0$ for the formal mating $g = P \sqcup Q$ of postcritically finite quadratic polynomials $P$ and $Q$, with $p$ and $q$ not in conjugate limbs of the Mandelbrot set. Moreover, assume that the combinatorial mating $f$ has not type* (2, 2, 2, 2)*.*

1. *If the formal mating $g$ has removable obstructions, it is essentially equivalent to the combinatorial mating $f$. The rational maps $f_n$ from the unmodified Thurston Algorithm converge to $f$. The images of marked points of $g$ collide according to their ray-equivalence classes under the iteration, and converge to marked points of $f$.*

2. *In both cases, when $g$ is combinatorially equivalent or essentially equivalent to $f$, consider the evolution of any periodic or preperiodic point $z$, which corresponds to a point in $\partial\mathcal{K}_p$ or $\partial\mathcal{K}_q$: then $x_n = \psi_n(z)$ converges to a periodic or preperiodic point of $f$. Different points are identified in the limit, if and only if they belong to the same ray-equivalence class.*

The second item is motivated by the videos of moving Julia sets, which are computed from the slow mating algorithm and meant to represent equipotential gluing [11]; it does not make sense when the formal mating is considered only up to isotopy with respect to postcritical points. There are two ways of understanding the statement in the context of the Thurston Algorithm:

- The formal mating $g$ is defined such that it is topologically conjugate to $P$ on the lower hemisphere and to $Q$ on the upper hemisphere. So there are subsets of the sphere corresponding to the Julia sets $\mathcal{K}_p$ and $\mathcal{K}_q$ and points corresponding to periodic and preperiodic points of $P$ and $Q$. Pick a homeomorphism $\psi_0$ and consider its lifts with $f_n \circ \psi_{n+1} = \psi_n \circ g$ in a suitable normalization. When $\widetilde{g} = g$, the Thurston Theorem 2.5 shows that the homeomorphisms $\psi_n$ converge up to homotopy with respect to the postcritical set of $g$, and $\psi_n(z)$ converges when $z$ is a marked point. So here the latter statement is extended to other points $z$ for the same sequence $\psi_n$ , not for any homotopic sequence.

- A finite number of these periodic or preperiodic points of $g$ may be marked in addition, giving a new pullback map on a higher-dimensional Teichmüller space. Then $\psi_n$ may be considered up to homotopy with respect to the marked set $Z$. If there are no collisions, the Thurston Theorem 2.5 gives convergence immediately, but Theorem 3.3 is needed in general.

**Proof:** Assume a normalization with critical points at 0 and $\infty$, and $1 = f_n(\infty)$ or the fixed point on the 0-ray is at 1. Actually, in the latter case we may mark $z = 1$ and the two $\beta$-fixed points in addition; so use the former normalization in the proof of item 1 and treat the second normalization as a special case of item 2.

1. According to [54], all obstructions of $g$ contain removable Lévy cycles, which consist of loops around periodic ray-equivalence classes with at least two marked points. A simple obstruction $\Gamma$ is obtained by adding all essential preimages, which are loops around preperiodic ray-equivalence classes containing at least two critical or postcritical points. Define the essential mating $\widetilde{g}$ by identifying all of these ray-equivalence classes, or alternatively, all disks bounded by $\gamma \in \Gamma$, to points. Then modify the map in neighborhoods of preimages containing at most one marked point as well. This is done without destroying the orbit of a single marked point within a disk; see Sections 2.5 and 3.1. Note that the original definition [54, 52] may involve collapsing a larger number of ray-equivalence classes with a single critical or postcritical point, but all possible choices of $\widetilde{g}$ are combinatorially equivalent. Now $g$, $\Gamma$, and $\widetilde{g}$ satisfy the assumptions of Proposition 3.7:

- Again by [54], $\widetilde{g}$ is unobstructed. Since it is not of type (2, 2, 2, 2), there is an equivalent rational map $f$, which defines the combinatorial mating and the geometric mating in fact.

- The critical points of $g$ are not identified in $\widetilde{g}$, because then $\widetilde{g}$ would not be defined properly as a branched cover of degree 2; this happens only when there are non-removable obstructions and the parameters are in conjugate limbs. No critical point is identified with 1 either, using a normalization different from $f(\infty) = 1$ when $f(\infty) = 0$.

- Loops bounding small tubular neighborhoods of disjoint simply connected ray-equivalence classes define disjoint disks, so $\Gamma$ is not nested.

- When a ray-equivalence class is not mapped homeomorphically, it contains a critical point of $P$ or $Q$, so it is preperiodic: periodic critical points are superattracting and not accessible by external rays. — Note that the first-return map of the disk around a periodic ray-equivalence class gives a homeomorphism of the corresponding piece, which is always finite-order and not pseudo-Anosov [22], since the postcritical points are connected by a tree mapped to itself.

So the formal mating $g$ is essentially equivalent to $f$, $\Gamma$ is the canonical obstruction, and Theorem 3.3 gives convergence of $f_n \to f$, and of colliding postcritical points as well. When $f$ is of type (2, 2, 2, 2), this statement is wrong in general [27].

2. Assume again that a postcritical point is normalized to 1, which is not in the same ray equivalence class as 0 or $\infty$, and the fixed point on the equator is marked in addition. Its convergence is obtained together with all marked points, and the normalization can be changed afterward to 1 by a rescaling with a convergent factor. — Given a finite number of periodic or preperiodic points of $P$ and $Q$, add all of their images and all ray-equivalent points, and consider the corresponding points in $\varphi_0(\mathcal{K}_p)$ and $\varphi_\infty(\mathcal{K}_q)$ together with the corresponding points on the equator of

$g$. Denote the union of postcritical ray-equivalence classes by $X$ and the additional classes by $Y$, set $X' = g^{-1}(X) \setminus X$ and $Y' = g^{-1}(Y) \setminus Y$. Now we have $X \cap Y = \emptyset$ and $X' \cap Y' = \emptyset$, but we may have $X' \cap Y \neq \emptyset$.

So there are finitely many disjoint ray-equivalence classes to consider. Each of these is a tree, since otherwise the topological and geometric matings would not exist. The essential mating $\widetilde{g}$ shall be defined by collapsing $X$ and modifying the new map in a small neighborhood of $X'$. The essential map $\widehat{g}$ for the larger set of marked points is defined by collapsing $X \cup Y$ and modification in a neighborhood of $X' \cup Y'$. Denote by $\Gamma$ the union of loops around the ray-equivalence trees in $X \cup Y$; it is a simple obstruction again.

When a homeomorphism $\psi_0$ is chosen and the Thurston pullback $f_n \circ \psi_{n+1} = \psi_n \circ g$ is applied, this gives the same homeomorphisms $\psi_n$ and rational maps $f_n$ as in item 1; these maps do not depend on the additional marked points, since the three normalized points are critical or postcritical. So the question is, do the homeomorphisms converge on the additional marked points, which follows when they converge in the larger Teichmüller space, i.e, up to homotopy with respect to the larger marked set. To apply Theorem 3.3 we only need to show that $\widehat{g}$ is unobstructed and equivalent to $f$. Otherwise for the larger set of marked points, the canonical obstruction of $g$ would contain a loop around several disks of $\Gamma$ or marked points of $g$.

- In the case without postcritical identifications in the formal mating, so $\widetilde{g} = g$, consider $\Psi_n$ from the proof of the Rees–Shishikura Theorem 4.2, which is defined by pulling back a specific homeomorphism $\Psi_0$ . Then $\Psi_n \to \Psi_\infty$ , which is a semi-conjugation mapping different ray-equivalence classes to different points. So the convergence claim is true for $\psi_0 = \Psi_0$ and $\widehat{g}$ is unobstructed.

- When $\widetilde{g} \neq g$, we have $\Psi_n = \widetilde{\Psi}_n \circ \pi_n \to \Psi_\infty$ , but $\Psi_0$ is not a homeomorphism. So consider $\widetilde{\Psi}_n$ instead, which are defined by a pullback with the essential mating $\widetilde{g}$. Since the essential map $\widehat{g}$ is defined by collapsing ray-equivalence classes of $g$ in $X \cup Y$ and modification around $X' \cup Y'$, an equivalent map can be defined as a component map from $\widetilde{g}$ as well. The limit of $\widetilde{\Psi}_n$ exists according to [12] and if $\widehat{g}$ was obstructed, then $\Psi_\infty$ would map different ray-equivalence classes to the same point.

Now Theorem 3.3 applies and gives convergence for any initial $\psi_0$ , in particular for slow mating and for equipotential gluing. See Remark 3.8 and [11] for questions of uniform convergence with respect to an arbitrary number of additional marked points. ∎

There are a few related ways to describe, which periodic or preperiodic points of $g$ converge to which point of $f$ :

- According to the Rees–Shishikura Theorem 4.2, there is a semi-conjugation $\Psi = \Psi_\infty$ from $g$ to $f$, which maps each ray-equivalence class to a unique point.

- Then $\Psi \circ \varphi_0$ and $\Psi \circ \varphi_\infty$ are partial semi-conjugations from $P$ on $\mathcal{K}_p$ or $Q$ on $\mathcal{K}_q$ to restrictions of $f$.

- $\gamma_f(\theta) = \Psi \circ \varphi_0 \circ \gamma_p(\theta) = \Psi \circ \varphi_\infty \circ \gamma_q(-\theta)$ is a semi-conjugation from the angle doubling map on $\mathbb{R}/\mathbb{Z}$ to $f$ on its Julia set.

## 5 Remarks on numerical implementations

Suppose $P$ and $Q$ are postcritically finite polynomials, not in conjugate limbs of the Mandelbrot set. Then the formal mating $g = P \sqcup Q$ is combinatorially equivalent or essentially equivalent to a rational map $f \cong P \coprod Q$. Consider the following implementations of the Thurston Algorithm for the formal mating, which should converge according to Theorem 4.3 except for numerical cancellations, unless $f$ is of type (2, 2, 2, 2):

**The medusa algorithm** was developed by Christian Henriksen and others under the guidance of John Hamal Hubbard [7]. Start with a Thurston map having marked points on two circles at specific angles; it is equivalent to the formal mating unless there are Misiurewicz parameters of satellite type — then the arguments from [26] give essential equivalence as well. A medusa is a graph connecting the images of marked points, which corresponds to the equator united with external rays from the equator to these points. Its pullback up to homotopy with rational maps provides a unique choice of preimages. Since this is an implementation of the Thurston Algorithm, the marked points and maps should converge, unless $f$ has type (2, 2, 2, 2).

However, medusa often seems to be numerically unstable even for simple examples: it begins to converge but after 50–100 iterations it oscillates wildly. It is not known whether this is a bug in the implementation, an unlucky choice of numerical parameters, or an unavoidable feature of this algorithm. I had expected that instability would be related to long ray connections converging to periodic points with a multiplier causing spiraling: then the equator would have to spiral as well and cannot be pruned to a homotopic curve with few long segments. But this idea was not confirmed by experiments; e.g., medusa did converge for $3/7 \sqcup 3/14$ and $12/31 \sqcup 19/62$, which have postcritical ray-equivalence classes of length four, and for $31/96 \sqcup 1/3$ and $511/1536 \sqcup 1/3$, which show significant spiraling. On the other hand, it diverged even in cases without postcritical identifications, e.g., $1/14 \sqcup 1/4$ and $19/60 \sqcup 1/3$. Note also that for the matings $5/28 \sqcup 13/28$ and $7/60 \sqcup 29/60$ of type (2, 2, 2, 2), the Thurston pullback accumulates on a four-cycle in configuration space according to [27]; medusa shows this behavior initially but oscillates after a few more iterations.

**Triangulations of the sphere** are used by Laurent Bartholdi in the GAP-package IMG [3, 16]. A Thurston map is represented algebraically as a biset [2], and it is easy to combine maps, as in a formal mating, or to apply a Dehn twist. Then a triangulation is constructed from the biset, which represents an isotopy class of homeomorphisms. It is pulled back to implement the Thurston Algorithm, with appropriate refinement and pruning.

When marked points get close to each other, the pullback is interrupted in the current version and an obstruction is searched instead, based on the assumption that points will be grouped in the observed way. In the case of formal matings with removable obstructions, this approach might be modified such that either the Thurston pullback is restarted with a component map, or such that the iteration is continued to allow a collapse of marked points according to Theorem 4.3. In the latter case, the pullback might become unstable, when a spiraling of marked points requires an excessive refinement of the triangulation.

**Slow mating** is much simpler to implement [23]. There are two basic ideas: pull back a path in moduli space, which encodes Teichmüller space implicitly. And use an initialization with $x_i \approx z_i/R$ for the marked points of $P$ and $x_i \approx R/z_i$ for the marked points of $Q$, where $R$ is a large radius. The resulting path provides an approximation to equipotential gluing [11]. — In the case of postcritical identifications, the path is not required to follow a spiraling equator. So there is a good chance to converge with just a small number of segments per marked point, and the algorithm will still be fast with a large number of segments. However, any discretization of a continuous path as a polygonal path should check, whether the exact pullback to piecewise arcs can be replaced homotopically with piecewise line segments again. This is easy in the case of quadratic polynomials, but more involved for quadratic rational maps [29]. Note that there may be a trade-off as well when using many small segments: homotopy violations shall happen less often, but detecting them may be numerically less stable.

**An initialization by angles** will be more convenient, but slow mating assumes that the parameters $p$ and $q$ are given as floating-point approximations. When angles are given instead, either run the spider algorithm [19] first to determine these parameters, or draw the parameter rays and improve the endpoints with Newton method. Alternatively, the slow mating algorithm can be modified such that the marked points are on two circles initially; the pullback would give the same marked points as medusa does, but be more stable.

**An encapture** according to [28] means that the critical value $\infty$ of $P$ is shifted to a preperiodic point $z_1 \in \partial\mathcal{K}_P$ along a dynamic ray. Again this Thurston map will be essentially equivalent to the geometric mating, possibly with different obstructions than the formal mating; the path in moduli space is initialized using an approximation to the dynamic ray.

So this paper suggests to treat removable obstructions by ignoring them, which is simple and fast: trust the slow mating algorithm to converge nevertheless. This route is taken naturally by equipotential gluing [11]. If you want to collapse ray-equivalence classes instead, you can determine the relevant angles recursively from the conjugate angle algorithm [26], but the topological part may be harder. When there are only direct connections between postcritical points of $P$ and $Q$, so a pseudo-equator exists [36], the modification can be done by taking a medusa with all points on the equator. Mary Wilkerson [56, 57] has an alternative implementation in this case: the pullback is controlled by a finite subdivision rule, which is constructed from Hubbard trees.

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