Slow mating and equipotential gluing

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1a. **Definition of matings**

Usually assume: $P(z) = z^2 + p$, $Q(z) = z^2 + q$ not in conjugate limbs of $\mathcal{M}$, with locally connected Julia sets.

Formal mating $g = P \sqcup Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is conjugate to P and Q on half-spheres. Sphere has images of Julia sets and rays, but natural conformal structure only on interior of Julia sets.

Topological mating $P \coprod Q$ shall be a branched cover $S^2 \to S^2$ obtained from g by collapsing all ray-equivalence classes.

Geometric mating shall be a rational map conjugate to the topological mating, $f \cong P \coprod Q$. The conjugation is conformal on the interior of the filled Julia sets.

1b. **Construction with Thurston theory**

Theorem (Rees–Shishikura–Tan)
Suppose P, Q are postcritically finite (PCF) and not in conjugate limbs (NCL). Then the topological mating $P \coprod Q$ and the geometric mating f exist.

Idea of the construction:

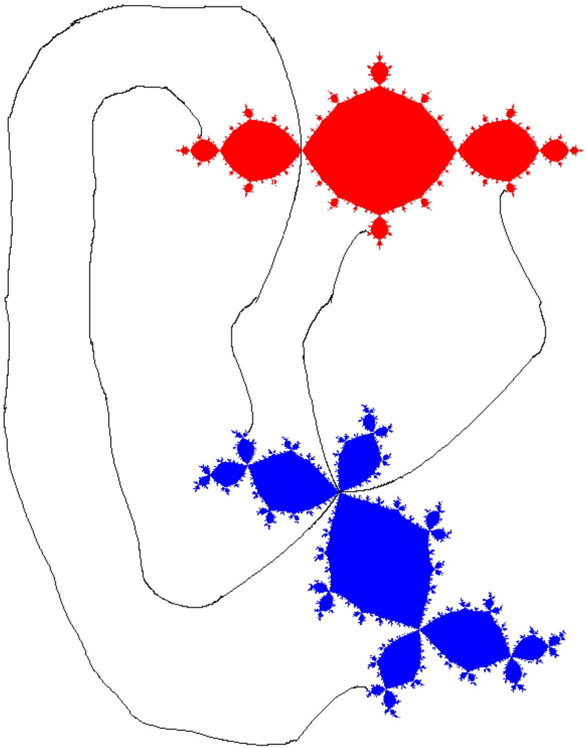
The formal mating $g = P \sqcup Q$ has either no obstructions or only removable obstructions, which surround postcritical ray-equivalence classes.

Modify g to an essential mating $\widetilde{g}$ by pinching these ray-equivalence classes. Then $\widetilde{g}$ is unobstructed.

If $\widetilde{g}$ is not of type $(2,\,2,\,2,\,2)$, the Thurston Theorem provides a combinatorially equivalent rational map f.

The Rees–Shishikura Theorem gives a semi-conjugation Ψ from g to f, which collapses all ray-equivalence classes to distinct points. So f is conjugate to $P \coprod Q$ and the latter exists. ■

1c. **Ray-equivalence classes**



1d. **Comparison of convergence properties**

For a Thurston map g, the pullback gives Homeomorphisms ψ_n and rational maps f_n. In Teichmüller space $\mathcal{T}$ there is a pullback map $\sigma_g : [\psi_n] \mapsto [\psi_{n+1}]$. Note different convergence statements in the case of matings:

For $\sigma_{\widetilde{g}}$, we have $[\psi_n] \to \tau$ and $f_n \to f$ for general ψ_0.

For σ_g, we have $\Psi_n \to \Psi$ for a special choice of Ψ_0, such that $f_n = f$.

Aim: For σ_g show that $\psi_n \to \Psi$ and $f_n \to f$ for more general ψ_0.

$$\begin{tikzcd}
S \arrow[r, "\psi_0"] & S_0 \\
S \arrow[u, "g"] \arrow[r, "\psi_1"] & S_1 \arrow[u, "f_0"'] \\
S \arrow[u, "g"] \arrow[r, "\psi_2"] & S_2 \arrow[u, "f_1"'] \\
S \arrow[u, "g"] \arrow[r, "\psi_3"] & S_3 \arrow[u, "f_2"'] \\
{} \arrow[u, "g"] \arrow[r, "\cdots"] & {} \arrow[u, "f_3"']
\end{tikzcd}$$

2a. **Equipotential gluing giving spheres**

Suppose $\mathcal{K}_p$ and $\mathcal{K}_q$ connected. Equipotential lines are curves with Boettcher coordinates $|\Phi_p(z)| = R$ or $|\Phi_q(z)| = R$, respectively. For $R > 1$, let U'_R and U_R be the domains bounded by equipotential lines of radius R and R^2, respectively. V'_R and V_R are analogous domains in the dynamic plane of Q.

The annuli in the exterior of the filled Julia sets are mapped to the round annulus $1/R < |z| < R$ by $\Phi_p(z)/R$ and $R/\Phi_q(z)$, respectively. Identify these annuli accordingly to obtain a Riemann surface; it is uniformized to the standard sphere $\mathcal{S}_R = \widehat{\mathbb{C}}$ by $u_R \cup v_R$.

We shall consider images of Julia sets, marked points, external rays of P and Q on $\mathcal{S}_R$.

Define maps $h_R : \mathcal{S}_\mathrm{e} \to \mathcal{S}_R$ corresponding to $z \mapsto z \cdot |z|^{\log R - 1}$. These are quasi-conformal with $K = \log R$ or $K = 1/\log R$. They form a holomorphic motion with respect to $\frac{\log R - 1}{\log R + 1}$ in the unit disk.

2b. **Equipotential gluing giving rational maps**

The rational map $f_R : \mathcal{S}_{\sqrt{R}} \to \mathcal{S}_R$ corresponds to the pair of polynomials $P : U_{\sqrt{R}} \to U_R$ and $Q : V_{\sqrt{R}} \to V_R$. As $R \to 1$, it is expected that f_R approximates the geometric mating under suitable conditions.

There is a quasi-regular map $g : \mathcal{S}_\mathrm{e} \to \mathcal{S}_\mathrm{e}$ such that $f_R \circ h_{\sqrt{R}} = h_R \circ g$.

Now g is conjugate to the formal mating on the filled Julia sets, and equivalent to it in the postcritically finite case. Then $[h_R]$ with $R = R_0^{2^{-t}}$ is a Thurston pullback.

2c. **Convergence properties of equipotential gluing**

Theorem (Chéritat–J.)
Suppose P and Q are postcritically finite and not from conjugate limbs. Assume $f \cong P \coprod Q$ is not of type $(2, 2, 2, 2)$. Then equipotential gluing with $R \to 1$ satisfies:

1. Marked points converge (with collisions) and $f_R \to f$.

2. The images of Julia sets, $\mathcal{J}_R = u_R(\partial\mathcal{K}_p) \cup v_R(\partial\mathcal{K}_q)$, satisfy $\mathcal{J}_R \to \mathcal{J}_f$ with respect to Hausdorff distance.

3. At least if P and Q are hyperbolic, the holomorphic motion converges to the semi-conjugation, $h_R \to \Psi$. So f is a conformal mating.

3a. **Convergence of marked points and rational maps**

If g is unobstructed, convergence is a direct application of the Thurston Theorem.

Suppose g has removable obstructions, so the pullback for g diverges in Teichmüller space and moduli space. Then show that marked points collide as expected and converge to common limits. See the general theorem on essential equivalence on page 3b.

It is based on the work of Selinger: he extended σ_g to augmented Teichmüller space $\widehat{\mathcal{T}}$, characterized canonical obstructions, and obtained an accumulation statement when the orbifold of a component map is not of type $(2, 2, 2, 2)$. The latter proof employs an interplay of different metrics and the product structure of the canonical boundary stratum.

Use this to construct a suitable path segment in moduli space, which is contained in a neighborhood of a point configuration with repetitions.

Finally, identify eigenvalues of the extended pullback map on configuration space (with additional arguments for identifications with critical points).

3b. **General convergence statement**

Theorem (J. 2016): *Suppose g is a bicritical Thurston map of degree d, and Γ is a completely invariant multicurve such that:*

- *All components of $S \setminus \Gamma$ except C are disks; the periodic disks are mapped homeomorphically.*
- *C is mapped to itself with degree d; the essential map $\widetilde{g}$ is equivalent to a rational map f.*

Using a normalization at 0, 1, ∞, such that no disk contains two of them, the Thurston Algorithm for the unmodified map g satisfies:

- *The curves of Γ are pinched, so $[\psi_n]$ diverges in $\mathcal{T}$.*
- *If f is not of type $(2, 2, 2, 2)$, we have $f_n \to f$. The marked points converge to (pre-)periodic points of f, identified according to the disks.*

3c. **Convergence on Fatou components**

Let $R_t = \mathrm{e}^{2^{-t}}$ and write f_t for f_{R_t}. Suppose the critical point 0 of P is k-periodic. Then $f_t \circ f_{t+1} \circ \cdots \circ f_{t+k-1}$ has a superattracting fixed point at 0.

Construct a small trapping region around 0, then $\psi_t = h_{R_t}$ converges to a composition of Boettcher conjugations there.

A finite pullback gives locally uniform convergence on every Fatou component of $u_{\mathrm{e}}(\mathcal{K}_p)$, and analogously for Q.

3d. **Convergence of Julia sets**

Claim: *On any finite set of repelling periodic or preperiodic points, $\psi_t(z)$ converges pointwise.*

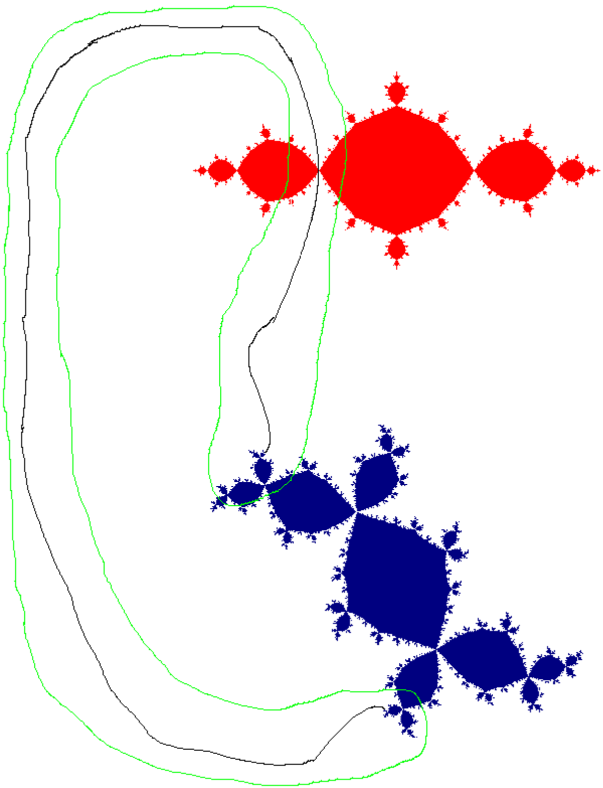
Basic idea: use additional obstructions as a tool to prove convergence of additional marked points, cf. page 3e.

To apply the general convergence result from page 3b, we need to show that the component map $\widehat{g}$ is unobstructed and equivalent to f (with additional marked points). Either extend Tan Lei's techniques, or use the semi-conjugation Ψ from g to f according to the Rees–Shishikura Theorem.

To show that $\mathcal{J}_f$ is in an ε-neighborhood of $\mathcal{J}_t$, cover it with a finite collection of $\varepsilon/2$-neighborhoods of repelling periodic points, and employ the claim above.

To show that $\mathcal{J}_t$ is in an ε-neighborhood of $\mathcal{J}_f$, use convergence on Fatou components, together with the fact that there are only finitely many components of diameter $\geq \varepsilon$.

3e. **Additional marked points and obstructions**



3f. **Convergence of holomorphic motions**

Here suppose in addition that both P and Q are periodic, so that the orbifold metric ρ_f is expanding for the perturbed maps f_R as well when $R \to 1$. (For a smaller expansion constant $1 < \lambda' < \lambda$. Here a tiny neighborhood of the critical points, well within the trapping region, is excluded.)

Now consider the ρ_f-length of segments $\{h_{R_t}(z) \,|\, T \le t \le T+1\}$ for z outside of the trapping region. It shrinks exponentially with T, uniformly in z. (Note that the estimate requires two terms, one involving the expanding property of the metric, and one coming from f_t depending on t.)

This gives uniform convergence of h_{R_t} as $t \to \infty$, or $R \to 1$, with respect to the ρ_f-metric and the spherical metric. The limit is a semi-conjugation from g to f, which coincides with Ψ from the Rees–Shishikura Theorem.

4a. **Slow mating**

For postcritically finite polynomials P and Q with critical orbits (p_i) and (q_i), the unmodified Thurston Algorithm for the formal mating $g = P \sqcup Q$ can be implemented with a path in moduli space as follows: Fix $R_1 \ge 5$ and interpolate the radius as $\log(R_t) = 2^{1-t} \log(R_1)$ for $0 \le t \le 1$. Set

$$\begin{aligned} x_i(t) &= \frac{1+(1-t)q/R_1^2}{1+(1-t)p/R_1^2} \cdot \frac{p_i/R_t}{1+(1-t)q/R_1^4\,(p_i-p)} \approx \frac{p_i}{R_t} \quad \text{and} \\ y_i(t) &= \frac{1+(1-t)q/R_1^2}{1+(1-t)p/R_1^2} \cdot \frac{R_t\Big(1+(1-t)p/R_1^4\,(q_i-q)\Big)}{q_i} \approx \frac{R_t}{q_i} \ . \end{aligned}$$

The initial path for $0 \le t \le 1$ can be pulled back uniquely for $1 < t < \infty$, choosing the sign below by continuity.

$$z_i(t+1) = \pm\sqrt{\frac{1-y_1(t)}{1-x_1(t)} \cdot \frac{z_{i+1}(t)-x_1(t)}{z_{i+1}(t)-y_1(t)}} \quad \text{for} \quad t \ge 0 \ .$$

4b. **Slow mating approximates equipotential gluing**

Suppose $\mathcal{K}_p$ and $\mathcal{K}_q$ are connected. So p and q may be postcritically infinite, in conjugate limbs, non-locally connected, or giving type $(2,\,2,\,2,\,2)$.

Proposition: *As $R \to \infty$, we have $u_R(z) \sim z/R$ and $v_R(z) \sim R/z$ on U'_R and V'_R, respectively.*

Proof by a composition of quasi-conformal maps with small dilatation.

Corollary: *For all $t \geq 0$, compare slow mating at time t to equipotential gluing at radius $R = R_0^{2^{-t}}$. Then h_R is uniformly close to slow mating in the sense of $\mathcal{T}$, $\widehat{\mathcal{T}}$, for a suitably initialized ψ_t, or considering marked points in $\widehat{\mathbb{C}}^{|Z|-3}$.*

In the postcritically infinite case, slow mating makes limited sense here as an approximation for a finite time with a large number of marked points.

5. **Non-convergence for Lattès maps of type** $(2,\,2,\,2,\,2)$

Thurston maps and rational maps of Lattès type $(2,\,2,\,2,\,2)$ are described by an affine map with an integer matrix, when the sphere is covered by the plane. There are 30 pairs of P, Q from non-conjugate limbs (9 types), such that the essential mating $\widetilde{g}$ obtained from $g = P \sqcup Q$ is of type $(2,\,2,\,2,\,2)$; it is equivalent to a rational Lattès map in each case.

Theorem (Chéritat, J.): *Now the Thurston Algorithm for g does not give $f_n \to f$ for every initial ψ_0, except for $\pm 1/4 \sqcup \pm 1/4$. There is a local attracting center manifold in configuration space. A typical orbit will spiral toward this manifold.*

So equipotential gluing and slow mating do not converge, unless they start on the stable manifold by accident. Numerical experiments suggest that this accident does not happen.

6. **Partial result for postcritically infinite matings**

It was or is hoped that equipotential gluing gives results about existence of geometric matings in postcritically infinite cases, and possibly a definition in non-locally connected cases (where the topological mating is undefined). However there is no general convergence proof for h_R or f_R.

The negative result for Lattès matings may be seen as making such a result less likely. There are a few positive partial results:

Remark: *When $q = 0$, h_R is just an affine rescaling, which sends a suitable point on the 0-ray of P to 1. It converges even when $\mathcal{K}_p$ is non-locally connected.*

Proposition: *When $\mathcal{K}_p$ and $\mathcal{K}_q$ are connected, the average length of rays in the spherical metric is $\overline{L} \le 2\sqrt{\log R} \to 0$.*

Note that this applies to postcritically infinite and non-locally connected cases, as well as to Lattès matings and to conjugate limbs. Proof with double integral and Chauchy–Schwarz, analogously to the Fatou proof that almost all rays land.

7. **Conjugate limbs — renormalized matings**

Suppose c is a center parameter of period $k > 1$, $p = c * p'$ and $\overline{q} = c * \overline{q'}$ with p', q' postcritically finite and not in conjugate limbs. Then p and q are in conjugate limbs. Suppose $f \cong P' \coprod Q'$ is not of type $(2,\, 2,\, 2,\, 2)$.

Conjecture: *Then the Thurston Algorithm for $g = P \sqcup Q$, in particular slow mating or equipotential gluing, gives $f_t \circ f_{t+1} \circ \cdots \circ f_{t+k-1} \to f$.*

When p or $\overline{q}$ is behind the small Mandelbrot set at c, replace them with tuned β-type Misiurewicz points.