Hurwitz equivalence and quadratic Lattès maps

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Abstract

In recent years, Thurston maps are discussed in terms of Dehn twists, Hurwitz equivalence, moduli space maps, and algebraic descriptions. For general quadratic Thurston maps, results of Koch [endo] are obtained here using quite elementary techniques. An example of twisted Lattès maps illustrates analytic and algebraic techniques as well, and the virtual endomorphism of the pure mapping class group is described.

1 Introduction

Acknowledgment: Several colleagues have contributed to this work by inspiring discussions and helpful suggestions. I wish to thank in particular Laurent Bartholdi, Arnaud Chéritat, Sarah Koch, Russell Lodge, Michael H. Mertens, Daniel Meyer, Volodymyr Nekrashevych, Kevin Pilgrim, Mary Rees, and Dierk Schleicher.

2 Thurston maps

Concepts like twisted rational maps and Hurwitz equivalence, moduli space maps and virtual endomorphisms, are discussed both for quadratic maps in general, and for Lattès maps in particular.

2.1 Combinatorial equivalence

2.2 Dehn twists and the pure mapping class group

2.3 Virtual endomorphisms

For a Thurston map g with postcritical set P, $|P| \geq 4$, consider the Teichmüller space $\mathcal{T}$, the projection $\pi : \mathcal{T} \to \mathcal{M}$ to moduli space, and the pullback map $\sigma_g : \mathcal{T} \to \mathcal{T}$. According to Section **??**, the pure mapping class group G acts on $\mathcal{T}$, and $\mathcal{M}$ is isomorphic to $\mathcal{T}/G$. Define the subgroup $H < G$ of liftable homeomorphisms: $h \in H$ if there is an $h' \in G$ with $g \circ h' = h \circ g$. Consider the virtual endomorphism $\Phi_g : H \to G$, $h \mapsto h'$, and the Hurwitz space $\mathcal{W} = \mathcal{T}/H$. Then $\pi = \pi_2 \circ \pi_1$ with covering maps $\pi_1 : \mathcal{T} \to \mathcal{W}$ and $\pi_2 : \mathcal{W} \to \mathcal{M}$. Moreover, $\mathcal{W}$ is represented by triples of rational maps f_τ and marked points in its domain and range, so there is a $\widetilde{\sigma}_g : \mathcal{W} \to \mathcal{M}$ with $\widetilde{\sigma}_g \circ \pi_1 = \pi \circ \sigma_g$ [DH, book2H, endo, KPS]. (An alternative notation is $\omega = \pi_1$, $Y = \pi_2$, $X = \widetilde{\sigma}_g$.) Classical and recent applications of these concepts include the following:

- τ, $\tau' \in \mathcal{T}$ satisfy $\pi(\sigma_g(\tau)) = \pi(\sigma_g(\tau'))$ and normalized $f_\tau = f_{\tau'}$, if and only if $\tau' = h \cdot \tau$ with $h \in H$. Since H has finite index in G and π_2 is finite, a family f_τ with $\pi(\tau)$ in a compact subset of $\mathcal{M}$ is compact. This fact is used in the proof of the Thurston Theorem for hyperbolic orbifolds [DH, book2H, teich], and in the Selinger proof of the Pilgrim Conjecture [ext, quadJung].

- Suppose g and $\widetilde{g}$ have the same postcritical set $P = \widetilde{P}$ (after conjugating one with a homeomorphism). Then they are Hurwitz equivalent in the sense of Definition **??**, if and only if $\mathcal{W} = \widetilde{\mathcal{W}}$ [endo].

- A moduli space map $k : \pi(\sigma_g(\mathcal{T})) \to \mathcal{M}$, as defined in Theorem **??**, exists when $\widetilde{\sigma}_g$ is injective [endo]; under additional assumptions, k extends to a critically finite endomorphism of $\mathbb{P}^{|P|-3}$.

- Various possible mapping characteristics of σ_g , like surjectivity or covering behavior, are obtained explicitly from properties of $\widetilde{\sigma}_g$ or k [BEKP, KPS].

- The Thurston Algorithm can be implemented by pulling back a path in moduli space; see Section 2.3 in [quadJung]. When k exists, this is a pullback with k, which may be understood in relation to the Julia set of k [BN].

- The Twisted Rabbit Problem was solved by Bartholdi–Nekrashevych [BN], who showed that some extension of the virtual endomorphism Φ_R is contracting on G; i.e., there is a finite absorbing set. See also Section **??** and [Lodge].

- If Γ is a multicurve and $h \in H$ a multitwist about Γ, then $\Phi_g : h \mapsto h'$ is related to the Thurston matrix of Γ. So g is unobstructed, if Φ_g is contracting on certain subgroups, i.e., the contraction coefficient is < 1 [Tw].

- For a postcritically finite rational map f, consider the iterated pullback of essential simple closed curves γ. Then f may have the property that all curves or multicurves become inessential or fall into a finite global attractor; this may be obtained from algebraic contraction properties of the virtual endomorphism $\Phi_f : H \to G$ [Tw, KPS]. The attracting property has applications both to the boundary behavior of the extended σ_f on augmented Teichmüller space $\widehat{\mathcal{T}}$ [ext, Lodge], and possibly to searching invariant multicurves.

2.4 Twisted rational maps

For a Thurston map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, the pure mapping class group G consists of isotopy classes of homeomorphisms, which are fixing the postcritical set P or the marked set Z pointwise. It is acting properly discontinuously on Teichmüller space $\mathcal{T}$ by $h \cdot \tau = [\psi \circ \varphi^{-1}]$ when $h = [\varphi] \in G$ and $\tau = [\psi] \in \mathcal{T}$. Actually this is a deck transformation for the cover $\pi : \mathcal{T} \to \mathcal{M}$ of moduli space, and $\mathcal{M} = \mathcal{T}/G$. Since the relation between $\mathcal{T}$ and $\mathcal{M}$ is fundamental for the Thurston pullback, many properties of σ_g are related to G. Moreover, mapping classes can be used to define and to relate different Thurston maps by pre- and postcomposition, which provides the algebraic structure of bisets and associated invariants [bookN]. A classical problem is to apply a Dehn twist to the Rabbit polynomial $f_R(z) = z^2 + c_R$ or to the Misiurewicz polynomial $f_{\mathrm{i}}(z) = z^2 + \mathrm{i}$ and to determine its combinatorial equivalence class [DH, algform]; it was solved by Bartholdi–Nekrashevych [BN].

A Dehn twist about a simple closed curve C, with at least two marked points in each complementary component, is a homeomorphism or isotopy class of homeomorphisms, which are the identity except in a neighborhood of C. A curve crossing C is mapped to a curve following C for a few rounds before continuing its way. This may be visualized as follows when C is round: let a neighborhood of C consist of some soft material and shear it by turning the inner disk around a few times, so that the marked points resume their original position. Note that turning counterclockwise gives a right twist, i.e., curves approaching C are deflected to the right. Dehn and Lickorish have shown that G is generated by a finite number of Dehn twists; see the references in [book1].

Example 2.1 (Twisting the Rabbit numerically)
There are three polynomials $f_c(z) = z^2 + c$, such that the critical value c is 3-periodic: the Rabbit, Corabbit, and the Airplane. In the dynamic plane of the Rabbit, let C be the obvious curve separating $z_1 = c$ and $z_2 = c^2 + c$ from $z_3 = 0$, and denote a simple right Dehn twist about C by φ. Now the problem is to determine the combinatorial equivalence class of $g_m = \varphi^m \circ f_R$ for $m \in \mathbb{Z}$; by the Lévy–Bernstein Theorem [book2], g_m is unobstructed, so it will be either c_R, c_C, or c_A.

*This problem is solved numerically by specifying an initial path in moduli space and pulling it back; see also Example **??**. We may either move z_1 around z_2 or vice versa, making m rounds counterclockwise. To reduce the dimension, it is convenient to rescale the plane such that $z_1 = 1$, so we must move z_2. Denoting $z_2 = u(t)$ we have an explicit rotation around $z_1 = 1$ for $0 \le t \le 1$, and the pullback*

$$u(t+1) = \frac{1}{\sqrt{1 - u(t)}} \tag{1}$$

for $t+1 > 1$, choosing the branch of the square-root such that the path is continuous. Looking at integer times, this gives $u(n) = \pi(\sigma_g^n(\tau_0))$ with $\tau_1 = [1]$. The point will stay at $c_R + 1$ for a few iterations, then jump away and converge to $c_A + 1$, $c_C + 1$, or back to $c_R + 1$.

For small values of $|m|$, the combinatorial equivalence class may be determined by finding curves homotopic to external rays explicitly, or by computing the iterated monodromy group of g. This gives $\varphi^1 \circ f_R \sim f_C$ and $\varphi^{-1} \circ f_R \sim f_A$ in particular. Bartholdi–Nekrashevych gave a solution for all $m \in \mathbb{Z}$ by using the virtual endomorphism $\Phi_R : H \to G$ of the pure mapping class group; see Section **??** and Remark **??**.3. They extended it to a contracting map $\overline{\Phi}_R : G \to G$ such that $h \circ f_R \sim \overline{\Phi}_R(h) \circ f_R$; after finitely many applications of $\overline{\Phi}_R$, every $h \in G$ becomes either φ^1, φ^{-1}, or 1. Moreover, they introduced the pullback of a path, which was interpreted as a pullback of $u(t) = k(u(t+1))$ with the moduli space map $u = k(u') = 1 - 1/u'^2$. The pure mapping class group G of f_R is described by its action on the iterated monodromy group of f_R, and it is isomorphic to the iterated monodromy group of k.

Remark 2.2 (Interactive implementation)
For interactive features related to the Thurston Algorithm, you may download and compile both Mandel 5.12 and Mandel 5.14 from www.mndynamics.com .

5.12 *The spider algorithm with legs, or alternatively with a path, is available for quadratic polynomials. Dehn twists and recapture surgery can be specified by dragging the critical value with the mouse.*

5.14 *Matings are available in the rational family 5.2, where ∞ is 2-periodic. Make your own movies of slow mating with the key F9.*
In cases with a one-dimensional moduli space map, specify it in family 5.0. Draw a path with the mouse and pull it back, choosing the keys "a" or "b" such that the path is appended. We have $k_R(u') = 1-1/u'^2$, $k_{\mathrm{i}}(u') = 1-2/u'^2$, and $k_{1/4}(u') = (u'^2-1)/(u'^2+1)$; the latter map is obtained with "q".
The pullback is useful to determine an iterated monodromy group as well.

Later versions shall combine and extend all of these commands, including more general matings as well as captures.

2.5 Hurwitz equivalence and moduli space maps

As explained after Example **??**, the family of twisted rabbits $\varphi^m \circ f_R$, $m \in \mathbb{Z}$, gives all 3-periodic polynomials R, C, A infinitely often. Consider the parameters c and $\widetilde{c}$ in the 1/3-sublimb of the period-2 component, which are defined by the internal address 1–2–6–7–13–14 and 1–2–6–7–9–14, respectively [BS, LS]. If φ is a suitable Dehn twist about the postcritical points $z_1 = c$ and z_{10} of f_c, the family $\varphi^m \circ f_c$, $m \in \mathbb{Z}$, is equivalent to two maps only, f_c and $f_{\widetilde{c}}$. This phenomenon will be discussed in terms of recapture surgery and with algebraic methods in [polyJung]; see also www.mndynamics.com/papers/goettingen11.pdf .

These two examples motivate the following question: given two rational maps f and $\widetilde{f}$ with the same branch portrait (mapping scheme), is there a φ in the pure mapping class group G of f, such that $g = \varphi \circ f$ is combinatorially equivalent to $\widetilde{f}$? It does not matter whether G acts from the left or right or both here, since all three cases would be topologically conjugate. Note that $\widetilde{f} = \psi_0 \circ (\varphi \circ f) \circ \psi_1^{-1}$ can be understood as $\widetilde{f} = (\psi_0 \circ \varphi) \circ f \circ \psi_1^{-1}$ as well, with $\psi_0 \circ \varphi = \psi_1$ on P but the two homeomorphisms not isotopic with respect to P. So the question is answered affirmatively, if and only if f and $\widetilde{f}$ are Hurwitz equivalent in the sense of the following definition. According to Theorem **??**, this is always the case for quadratic maps. Related concepts have further applications to mapping properties of σ_f, e.g.; see Section **??**.

Definition 2.3 (Branch portrait and Hurwitz equivalence)
Suppose g is a Thurston map of degree $d \ge 2$ with critical set Ω and marked set Z, so that $g(\Omega \cup Z) \subset Z$. Let $\widetilde{\Omega}$ and $\widetilde{Z}$ be corresponding sets for $\widetilde{g}$.

1. *The **branch portrait** of g is a directed graph describing the action of $g : \Omega \cup Z \to Z$, with multiple arrows from critical points to critical values. So the branch portraits of g and $\widetilde{g}$ are **isomorphic**, if there is a bijection $\psi : \Omega \cup Z \to \widetilde{\Omega} \cup \widetilde{Z}$ with $\psi \circ g = \widetilde{g} \circ \psi$ on $\Omega \cup Z$, preserving the local degree of critical points (multiplicity of arrows).*

2. *Two Thurston maps g, $\widetilde{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are (**pure**) **Hurwitz equivalent**, if there are homeomorphisms ψ_0, $\psi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\psi_0 \circ g = \widetilde{g} \circ \psi_1$, such that $\psi_1 = \psi_0$ on Z.*

Note that the branch portrait is not surjective, when there are preperiodic marked points that are not postcritical. More precisely, we have defined a dynamic portrait, but since we do not iterate here, it is used as a static portrait. This means that the sets Z in the domain and range of g might as well be different. — Hurwitz equivalence is a weaker notion than combinatorial equivalence, because ψ_0 and ψ_1 need not be isotopic. When $(\psi_0 \,.\, \psi_1)$ gives a Hurwitz equivalence and $\omega \in \Omega$, then $\psi_1(\omega)$ is a critical point of $\widetilde{g}$, but $\psi_0(\omega)$ may be arbitrary, unless $\omega \in Z$. When g and $\widetilde{g}$ are Hurwitz equivalent, then $\psi_1 \,|\, (\Omega \cup Z)$ defines an isomorphism ψ of branch portraits. The converse is true in the quadratic case:

Theorem 2.4 (Hurwitz equivalence, following Koch)
Suppose that $g,\, \widetilde{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *are quadratic Thurston maps with critical sets* Ω, $\widetilde{\Omega}$ *and marked sets* Z, $\widetilde{Z}$. *If there is an isomorphism* $\psi : \Omega \cup Z \to \widetilde{\Omega} \cup \widetilde{Z}$ *of branch portraits, then it extends to a Hurwitz equivalence* $\psi_0 \,,\, \psi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $\psi_0 \circ g = \widetilde{g} \circ \psi_1$, *with* $\psi_0 = \psi$ *on* Z *and* $\psi_1 = \psi$ *on* $\Omega \cup Z$.

The branch portrait of g may be symmetric, i.e., there is an automorphism interchanging the critical orbits. In this case, we have a Hurwitz equivalence from g to g interchanging the critical points, or two different Hurwitz equivalences from g to $\widetilde{g}$. The theorem extends to bicritical maps of degree $d \ge 3$, if the cyclic displacement of certain points is the same for g and $\widetilde{g}$. E.g., the unicritical polynomials $cz^4 + 1$ with preperiod 1 and period 1 form three Hurwitz equivalence classes, which are given by $c = -2$, $c = -1 + \mathrm{i}$, and $c = -1 - \mathrm{i}$.

Theorem **??** was not stated formally in [endo]. Sarah Koch obtained it within the proof of her results on moduli space maps, see below. For this reason, the case of $\Omega \cap P = \emptyset$ was not included. On the other hand, she treats multicritical topological polynomials with $\Omega \subset P$ as well.

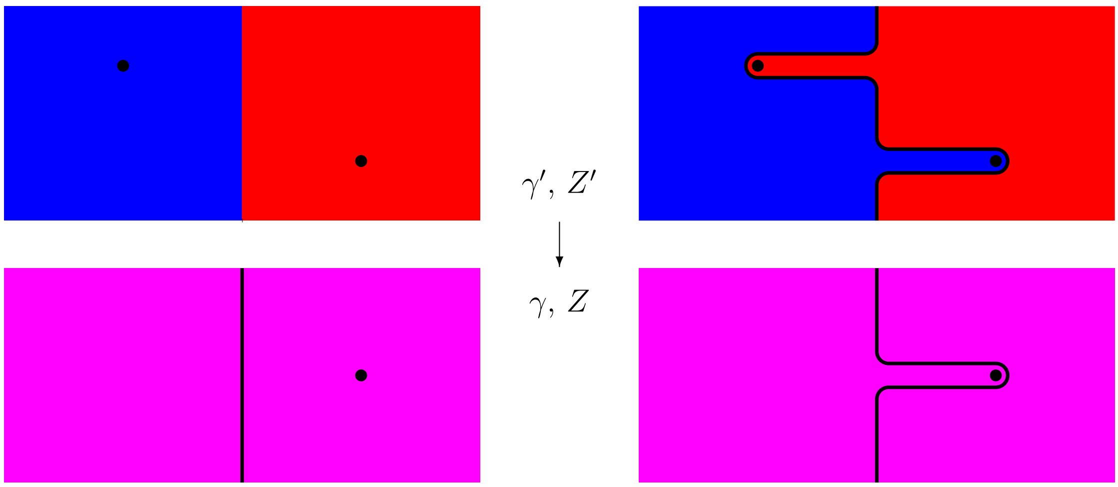

Figure 1: In the proof of Theorem **??**, the colors of z and $-z$ are exchanged by deforming the curve γ. This can be done by choosing an arc from an inner point of γ to $g(z) = g(-z)$ and modifying γ in a small neighborhood of this arc, so that other marked points are not affected. The preimage $\gamma' = g^{-1}(\gamma)$ is deformed such that z and $-z$ exchange their colors.

Proof of Theorem ??: Since g is a branched cover of degree 2, it has two distinct critical points and two distinct critical values, which may coincide with critical points. Denote the involution by a minus sign, such that $g^{-1}(g(z)) = \{z,\ -z\}$ for all $z \in \widehat{\mathbb{C}}$. Set $Z' = \Omega \cup Z \cup -Z \subset g^{-1}(Z)$ and analogously for $\widetilde{g}$, then ψ extends uniquely to a bijective semi-conjugation $\psi : Z' \to \widetilde{Z}'$. Choose a simple arc γ connecting the critical values of g and avoiding $Z \setminus g(\Omega)$. Then $\gamma' = g^{-1}(\gamma)$ is a simple loop through the critical points, whose complement consists of a red disk and a blue disk. Do an analogous construction for $\widetilde{g}$. Our aim will be to construct ψ_1 such that it maps the red disk of g to the red disk of $\widetilde{g}$.

The problem is that when $z \in Z'$ is red, $\psi(z)$ may be blue. So we need to modify our curves. Figure **??** illustrates the idea: in a finite number of steps, for each $z \in Z$ with incompatible colors, modify γ continuously so that it crosses $g(z)$ once, without crossing any other point in Z. Then $\gamma' = g^{-1}(\gamma)$ is deformed such that z and $-z$ exchange their colors, while every other point in $Z' \setminus \Omega$ keeps its color. After finitely many deformations, we can define ψ_0 and ψ_1. First let $\psi_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be any homeomorphism with $\psi_0 = \psi$ on Z and $\psi_0(\gamma) = \widetilde{\gamma}$. Then there are two possible choices for a homeomorphism $\psi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\psi_0 \circ g = \widetilde{g} \circ \psi_1$; choose it such that red goes to red.

Now for $z \in Z \setminus \Omega$ we have $\psi_1(z) \in \widetilde{g}^{-1}(\psi_0(g(z))) = \{\psi(z),\ -\psi(z)\}$. Since the curves and the homeomorphism have been constructed to respect the color, this gives $\psi_1 = \psi$ on $Z \setminus \Omega$, and by construction on Ω as well. — For bicritical maps with degree $d \geq 3$, there are d colors with a cyclic order invariant under deformations. If z_1, $z_2 \in Z$ with $g(z_1) = g(z_2)$, then $\psi(z_1)$, $\psi(z_2) \in \widetilde{Z}$ must have the same cyclic displacement; otherwise there is no choice of γ such that $\psi(z_1)$ has the same color as z_1 and $\psi(z_2)$ the same color as z_2. This condition is void if all points in $\Omega \cup Z$ are periodic. ∎

The Hurwitz space $\mathcal{W}$ of g gives an intermediate cover $\mathcal{T} \to \mathcal{W} \to \mathcal{M}$; see Section **??** for its definition and various applications. The Thurston pullback $\sigma_g : \mathcal{T} \to \mathcal{T}$ descends to a correspondence on moduli space, whose inverse often is a function k in fact. This moduli space map may extend to a critically finite endomorphism of $\mathbb{P}^{|P|-3}$. An explicit construction of k is discussed below. In Section 5.2 of [endo], Koch constructs Hurwitz spaces $\mathcal{W}$ under various assumptions. Hurwitz equivalence is verified from conjugation of Hurwitz spaces, and a moduli space map k is obtained when the second projection $\widetilde{\sigma}_g : \mathcal{W} \to \mathcal{M}$ is injective.

Theorem 2.5 (Moduli space map, following Koch)
*Suppose g is a quadratic Thurston map with $|P| \geq 3$, without additional marked points. When does a **moduli space map** exist, i.e., $k : \pi(\sigma_g(\mathcal{T})) \to \mathcal{M} = \pi(\mathcal{T})$ with $k \circ \pi \circ \sigma_g = \pi$?*

a) Assume at least one critical point of g is postcritical, i.e., periodic or in the orbit of the other critical point. Then there is a moduli space map $k : \pi(\sigma_g(\mathcal{T})) \to \mathcal{M}$.

b) Assume that no critical point is postcritical, i.e., they are both preperiodic and their orbits are either disjoint, or meet in non-critical points only. If the critical points are not marked, there is no rational moduli space map $k : \pi(\sigma_g(\mathcal{T})) \to \mathcal{M}$.

c) Assume again that no critical point is postcritical. Mark the critical points in addition and restrict the higher-dimensional spaces to "odd homeomorphisms" and "even rational maps." Then a rational moduli space map k exists in this setting, if and only if the two critical values of g are mapped to the same point. E.g., this applies to quadratic Lattès maps of case a) or b).

Proof: Recall that the Thurston pullback σ_g on Teichmüller space $\mathcal{T}$ is defined as follows: for $\tau = [\psi]$ there is a rational map f_τ and another homeomorphism ψ', such that $\psi \circ g = f_\tau \circ \psi'$, and then $\sigma_g(\tau) = [\psi']$. We are interested in the point configurations $x = \pi(\tau)$ with components in $\psi(P)$, $x = (x_1, \ldots, x_{|P|}) = (\psi(z_1), \ldots, \psi(z_{|P|}))$, and $x' = \pi(\sigma_g(\tau))$ with components $x_i' = \psi'(z_i)$. If $g(z_i) = z_j$, then $f_\tau(x_i') = x_j$, and in general x' is not determined by x alone: additional information from the isotopy class of ψ is required to determine ψ' and x'.

Now the question is to determine x from $x' \in \pi(\sigma_g(\mathcal{T}))$ without knowing τ. Moduli space defines point configurations up to Möbius conjugation, and it is represented as a subset of $\widehat{\mathbb{C}}^{|P|-3}$ by fixing the positions of three postcritical points. We shall assume that the critical values of $f = f_\tau$ are 0 and ∞, and $f_\tau(\infty) = 1$. The latter condition is satisfied by a rescaling unless $f(\infty) \in \{0, \infty\}$ — in that case, 0 and ∞ must be interchanged by conjugating with $z \mapsto 1/z$. With unknown critical points u and v, this gives

$$f(z) = f_\tau(z) = \Big(\frac{z-u}{z-v}\Big)^2 \ . \tag{2}$$

a) If both critical points are postcritical, then u and v are just components of x', so x' determines f and the components of x are explicit quadratic rational maps in the components of x'. If only one postcritical point is critical, the branch portrait contains a unique point with two preimages. So given x', we know two components α and β, such that $f(\alpha) = f(\beta)$. This gives the equation

$$\frac{\alpha-u}{\alpha-v} = -\frac{\beta-u}{\beta-v} \qquad \text{or} \qquad uv - \frac{\alpha+\beta}{2}(u+v) + \alpha\beta \ , \tag{3}$$

with suitable modification in the case of $\alpha = \infty$ or $\beta = \infty$. Now since either u or v is determined as a component of x', the other one is computed from (**??**).

b) If both critical points are not postcritical, there are four distinct components α, β, γ, δ of x' with $f(\alpha) = f(\beta)$ and $f(\gamma) = f(\delta)$, so u and v must satisfy (**??**) and the analogous equation with γ and δ. This system has two solutions, related by interchanging the values of u and v. For each x' the pullback relation on moduli space is satisfied by two different functions f, and in general two different configurations x. Morally, this means: replacing ψ with $1/\psi$ and f_τ with $1/f_\tau$ would give the same x'. With full information about ψ we know that $1/\psi$ does not map specific critical values of g to 0 and ∞, but given x' we cannot distinguish between $x = (x_i)$ and $(1/x_i)$. These are not equal since $\psi(P) \setminus \{0, \infty\} \not\subset \{1, -1\}$ for any or all $[\psi_0] \in \mathcal{T}$. — This interpretation does not seem to be a complete proof, however, because x and the inverse could be given by distinct maps on moduli space, one of which would be correct. So let us consider the explicit solution of the system given by (**??**) and the other equation with γ and δ:

$$u,\, v = \frac{(\alpha\beta - \gamma\delta) \pm \sqrt{(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)}}{\alpha+\beta-\gamma-\delta} \ , \tag{4}$$

which reduces to, e.g., $u,\, v = \beta \pm \sqrt{(\beta-\gamma)(\beta-\delta)}$ if $\alpha = \infty$. Note that in any case, at most three of the components α, β, γ, δ of x' are fixed values 0, 1, ∞, and at least one is a free variable x_i'. Thus the radicand does not consist of squares only and u, v are not defined by single-valued functions of the x'_i. Then choosing a component x'_i of x' such that $f(x'_i) \neq 1$, the component $f(x'_i)$ of x will not be a rational function of x'. — With some effort, it should be possible to construct a closed path in $\pi(\sigma_g(\mathcal{T}))$ such that the radicand in (**??**) follows a closed path around 0; then there will be no algebraic moduli space map either.

c) Under the same assumptions on the Thurston map g, treat the critical points as additional marked points, which increases the dimension of $\mathcal{T}$ and $\mathcal{M}$ by two. Then the range of σ_f is contained in an invariant submanifold of codimension two, such that ψ maps a specific critical point of g to 0 and the other one to ∞, and such that $\psi(\widetilde{z}) = -\psi(z)$ whenever $g(\widetilde{z}) = g(z)$ and $\widetilde{z} \neq z$. For the example of Lattès maps, see Section **??**.3. Note that this normalization affects both the domain and range of f: for indices $i \neq j$ with $g(z_i) = g(z_j)$, we have not only $x'_i = -x'_j$ but also $x_i = -x_j$. We have parameters a, b and components α, β, γ, δ of x' with distinct squares, such that

$$f(z) = \frac{z^2 + a}{z^2 + b} \; , \qquad f(\pm\alpha) = -f(\pm\beta) \quad \text{and} \quad f(\pm\gamma) = -f(\pm\delta) \; . \tag{5}$$

Solving this system of equations for a and b gives, analogously to (**??**),

$$a,\, b = \frac{(\alpha^2\beta^2 - \gamma^2\delta^2) \pm \sqrt{(\alpha^2 - \gamma^2)(\alpha^2 - \delta^2)(\beta^2 - \gamma^2)(\beta^2 - \delta^2)}}{\alpha^2 + \beta^2 - \gamma^2 - \delta^2} \; . \tag{6}$$

Again at most three of the four x'_i are fixed values, and the radicand is not a square if none of these is ∞. If, e.g., $\alpha = \infty$, we have $a,\, b = \beta^2 \pm \sqrt{(\beta^2 - \gamma^2)(\beta^2 - \delta^2)}$, which is reduced to $a,\, b = \pm\gamma\delta$ only if $\beta = 0$. So we must have $f(\infty) = -f(0)$, or equivalently, $f^2(\infty) = f^2(0)$. This means that g maps both critical values to the same point as well. Finally, note that $a,\, b = \pm\gamma\delta$ gives two different single-valued candidates for a moduli space map k; by analytic continuation, one of these gives the correct (x_i) everywhere and the other one would give $(1/x_i)$ everywhere. ■

3 Lattès maps

3.1 Rational maps and Thurston maps of Lattés type

3.2 Lifting to an affine map

$$S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

3.3 Combinatorial equivalence and binary quadratic forms —Jointly with Michael H. Mertens

Equivalence $S^{-1}AS = \pm\widetilde{A}$ with a conjugator $S \in SL_2(\mathbb{Z})$ can be checked algorithmically. It is related to a classical problem in number theory in fact, the classification of integral quadratic forms. By work of Lagrange, Legendre, Gauß, and Zagier, e.g., there are effective reduction algorithms. — We shall use the normal forms

$$C_t^+ = \begin{pmatrix} 0 & -2 \\ 1 & t \end{pmatrix} \; , \qquad C_t^- = \begin{pmatrix} 0 & 2 \\ -1 & t \end{pmatrix} \; . \tag{7}$$

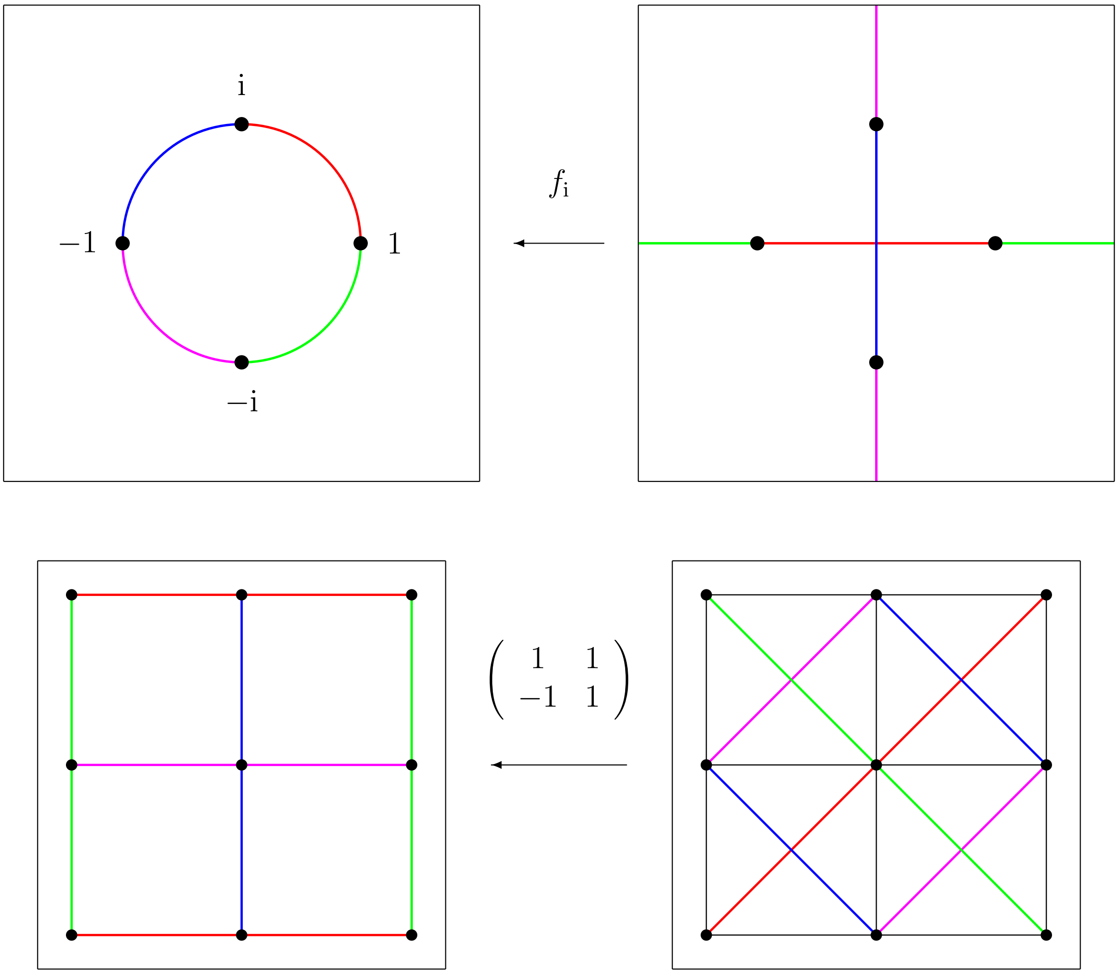

Figure 2: f_{i}.

C_t^+ is the companion matrix of trace $t = a + d$ and the second matrix is chosen such that A is conjugate to C_t^-, if $-A$ is conjugate to C_{-t}^+. So an idea to determine equivalence classes for some modulus $|t|$ of the trace would be to check whether every matrix is conjugate to $C_t^\pm$, and whether these two are conjugate. Both questions lead to representations by quadratic forms: are there $x,\, y \in \mathbb{Z}$ with

$$cx^2 + (d-a)xy - by^2 = \pm 1 \qquad \text{or} \qquad -x^2 + txy - 2y^2 = 1\,, \tag{8}$$

respectively? In both equations, the discriminant is $\Delta = t^2 - 8$. In simple cases, we may either find an explicit solution or see a contradiction, avoiding the general reduction theory.

Example 3.1 (Rational classes, jointly with Michael Mertens)
When $|t| \le 2$, then g is equivalent to a rational map. For each value of t, there are two conjugacy classes of matrices, and there are two classes up to sign with $|t| = 1$ or $|t| = 2$, respectively. The case of $t = 0$ is special: C_0^+ and C_0^- are not conjugate, but $C_0^- = -C_0^+$, so there is only one combinatorial equivalence class. Now $t = 2$ can be treated with the unique factorization of Gaussian integers $\mathbb{Z}[\mathrm{i}]$ as well: When A has $c > 0$, there are $k,\, m,\, n \in \mathbb{Z}$ with $k,\, m > 0$ and $km = n^2 + 1$, such that

$$A = \begin{pmatrix} 1-n & -m \\ k & 1+n \end{pmatrix} . \quad \text{Let} \quad \tilde{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \tag{9}$$

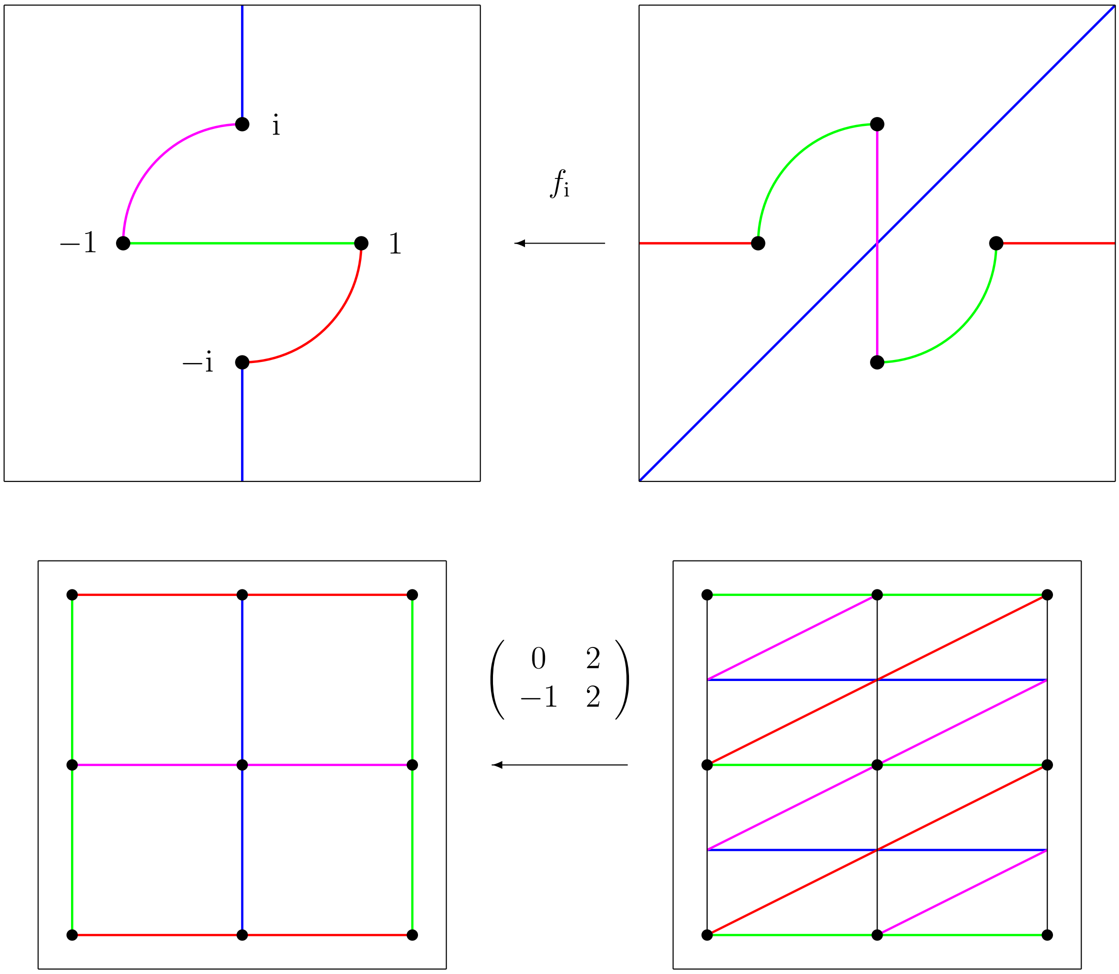

Figure 3: f_{i}.

The ansatz $AS = S\widetilde{A}$ *to determine* S *gives two equations, which are equivalent to a single complex equation* $k(\alpha + \mathrm{i}\beta) = -(n + \mathrm{i})(\gamma + \mathrm{i}\delta)$. *Now the prime factorization of* $km = (n+\mathrm{i})(n-\mathrm{i})$ *gives* $u,\ v \in \mathbb{Z}[\mathrm{i}]$ *with* $n+\mathrm{i} = uv$, $k = |u|^2$, *and* $m = |v|^2$. *Then* S *is obtained as* $\alpha + \mathrm{i}\beta = v$ *and* $\gamma + \mathrm{i}\delta = -\overline{u}$, *which has determinant* $\mathrm{Im}(uv) = 1$. *— In a way, this argument can be reverted, so that the Thurston Theorem implies that every divisor of* $n^2 + 1$ *is a sum of two squares.*

Example 3.2 (Non-rational classes, jointly with Michael Mertens)
When $|t| \geq 3$, g *is not equivalent to a rational map. For* A *of trace* $t = 3$, *we use* $-bc = (a-1)(a-2)$ *to show that* A *is diagonalized with* $S \in SL_2(\mathbb{Z})$. *For* $4 \leq t \leq 11$, *the class number varies between one and two; these can be determined online from* $\Delta = t^2 - 8$. *See www.numbertheory.org/php/classnopos0.html . For* $t = 12$ *there are four equivalence classes of quadratic forms, or conjugacy classes of matrices, and four combinatorial equivalence classes:*

$$\begin{pmatrix} 0 & \mp 2 \\ \pm 1 & 12 \end{pmatrix} \qquad \begin{pmatrix} 2 & \pm 6 \\ \pm 3 & 10 \end{pmatrix} \tag{10}$$

While these class numbers count primitve qudratic forms, so no prime number divides all three coefficients, it may happen in our case that $(d-a)$ *shares a common factor with* b *and* c. *This requires a prime* p *with* $p^2|(t^2-8)$, *and* $p = 2$ *is excluded* *by determinant 2. The first example is obtained for $t = 20$ and $p = 7$: there are only two classes of primitive quadratic forms with $\Delta = 392$, but at least four combinatorial equivalence classes. Checking (??) for the second matrix in (??) gives $\pm 7x^2 + 14xy \mp 7y^2 = 1$ or $= -1$, so it is not conjugate to a companion matrix.*

$$\begin{pmatrix} 0 & \mp 2 \\ \pm 1 & 20 \end{pmatrix} \qquad \begin{pmatrix} 3 & \pm 7 \\ \pm 7 & 17 \end{pmatrix} \tag{11}$$

4 Virtual endomorphisms and Lattès maps

While the pullback relation on curves has a finite global attractor, if f is a hyperbolic quadratic polynomial or a suitable preperiodic polynomial, this cannot happen for a Lattès map: $f^{-1}(P) = \Omega \cup P$ shows that the preimages of an essential curve are essential again. Moreover, these curves correspond to rational boundary points of the upper halfplane, which are pulled back with the Möbius transformation σ_f.

Theorem 4.1 (Virtual endomorphism and pullback of curves)
Suppose f is a quadratic rational Lattès map of type $(2, 2, 2, 2)$ without additional marked points. Then the liftable homeomorphisms form a normal subgroup $H < G$ with index 2 and the virtual endomorphism $\Phi_f : H \to G$ of the pure mapping class group is injective, not surjective, and not contracting.

Φ_f is represented by a linear automorphism of $\mathbb{Q}^{2\times 2}$, whose eigenvalues are related to the multiplier ρ of the Thurston pullback σ_f. For the cases a)–d) according to Sections ?? and ?? we have:

a) and b): $\Phi_f : H \to H$ is an isomorphism of finite order. Under Φ_f, every $h \in H$, $h \neq 1$, has strict period 2 in case a), period 4 in case b). Under iterated pullback, every essential simple closed curve is strictly 2-periodic or 4-periodic, respectively.

c) and d): Now H is not invariant under Φ_f, so the domain of Φ_f^n is shrinking. Every essential simple closed curve has an infinite orbit under iterated pullback.

Pullback of curves: The preimage of an essential curve consists of one essential curve or two homotopic essential curves. Lifting f to $\mathbb{R}^2/\mathbb{Z}^2$, it is represented by $\vec{x} \mapsto A\vec{x} + \vec{\kappa}$, and essential curves correspond to classes of parallel lines with rational slope, described by vectors in $\mathbb{Z}^2$. These are pulled back with A, so a homotopy class is n-periodic, if and only if it corresponds to an integer eigenvalue of A^n [ext]. Now in case a), A has complex eigenvalues and $A^2 = -2$. In case b), A and A^2 do not have integer eigenvalues, but $A^4 = -4$. In cases c) and d), the eigenvalue η^n would be integer if $\rho = 2/\eta^2$ satisfied $\rho^n = 1$, but ρ is not an algebraic integer since $\rho^2 + \frac{3}{2}\rho + 1 = 0$.

Computing the virtual endomorphism: All matrices are understood up to a change of sign. Isotopy classes of homeomorphisms in the pure mapping class group G of f are covered by affine maps of the form $\vec{x} \mapsto Q\vec{x}$, such that $Q \in SL_2(\mathbb{Z})$ has odd entries on the diagonal and even entries beside it [SY]; thus the postcritical set $\Lambda/2$ is fixed pointwise modulo $\Lambda = \mathbb{Z}^2$. Let us denote the matrix group by the same symbol G and recall that composition in G is written from right to left, which corresponds to matrices acting on vectors from the left. Now the lift of mapping classes according to $f \circ h' = h \circ f$ corresponds to $AQ' = QA$.

In contrast to f, the matrix A has a unique inverse, so we may consider the map $Q \mapsto Q' = A^{-1}QA$, which is a linear map $\mathbb{Q}^{2\times2} \to \mathbb{Q}^{2\times2}$ and an inner automorphism of $GL_2(\mathbb{Q})$. The subgroup $H < G$ is represented by matrices $Q \in G$ with $Q' \in G$, and $\Phi_f : Q \mapsto A^{-1}QA$ is the virtual endomorphism; it is injective by construction. Note that for $|P| > 4$, every injective virtual endomorphism of G would be given by a unique conjugation in G according to the Bell–Margalit Theorem 6.9 in [KPS]; in our case with $|P| = 4$, there is a conjugation in $GL_2(\mathbb{Q})$ but not in G.

According to Sections **??** and **??**, A has the eigenvalues η and $\overline{\eta} = 2/\eta$. The Thurston pullback σ_f has the derivative $\rho = 2/\eta^2$ at its fixed point. Now we consider $\mathbb{Q}^{2\times2}$ as a vector space $\mathbb{Q}^4$ and the map $Q \mapsto A^{-1}QA$ is a linear map in $GL_4(\mathbb{Q})$. Its "eigenmatrices" in $\mathbb{C}^{2\times2}$ are dyadic products, such that the first factor is an eigenvector of A^{-1} and the second factor is an eigenvector of A transposed. Thus the eigenvalues of $\Phi_f \in GL_4(\mathbb{Q})$ are the quotients $\eta/\eta = 1$, $\overline{\eta}/\overline{\eta} = 1$, $\overline{\eta}/\eta = \rho$, and $\eta/\overline{\eta} = \overline{\rho}$. The latter are both -1 in case a) and $\pm\mathrm{i}$ in case b).

More explicitly: The pure mapping class group is a free group generated by two right Dehn twists S and T, such that only T is supported on a curve separating the critical values, so T is not liftable by f but T^2 and S are [BN]. Let us consider the following concrete representations: the companion matrix A represents case a) for $t = 0$, case b) for $t = 2$, and cases c) and d) for $t = 1$; the complex conjugate cases with $t = -2$ and $t = -1$ would be related by orientation-reversing conjugations.

$$A = \begin{pmatrix} 0 & -2 \\ 1 & t \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix} \tag{12}$$

Now a short calculation gives the following expression for the virtual endomorphism Φ_f. Both Q and Q' must have odd entries on the diagonal and even entries beside it. Observing $4|(\delta - \alpha)$, the formula (**??**) shows that in each of the three cases, the same matrix subgroup $H < G$ is obtained: $Q \in G$ belongs to H, if and only if $4|\beta$. So T is not in H but S, T^2, and $U = T^{-1}ST$ are.

$$\Phi_f \ : \quad Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \mapsto \quad Q' = \begin{pmatrix} \delta + \frac{t}{2}\beta & \frac{t^2}{2}\beta + t(\delta - \alpha) - 2\gamma \\ -\frac{1}{2}\beta & \alpha - \frac{t}{2}\beta \end{pmatrix} \tag{13}$$

The relation $T^nS^mT^k = T^{n+1}U^mT^{k-1}$ shows that every $Q \in G$ can be written as w or wT, where w is a word in S, T^2, U and their inverses. So $H < G$ is generated by S, T^2, U; it is a normal subgroup of index 2. Writing $Q \in G$ as a word in $S^{\pm1}$, $T^{\pm1}$, it belongs to H if and only if the total number of $T^{\pm1}$ is even. Now Φ_f acts on the generators of H as follows:

$$\begin{aligned} &\text{a) } t = 0, \quad \Phi_f \ : \ S \mapsto T^2 \qquad T^2 \mapsto S \qquad U \mapsto S^{-1}T^{-1}S^{-1}T^{-1} \\ &\text{b) } t = 2, \quad \Phi_f \ : \ S \mapsto T^2 \qquad T^2 \mapsto U \qquad U \mapsto T^{-1}S^{-1}T^{-1}S^{-1} \\ &\text{c)d) } t = 1, \quad \Phi_f \ : \ S \mapsto T^2 \qquad T^2 \mapsto T^{-1}S^{-1} \qquad U \mapsto S^2 \end{aligned} \tag{14}$$

Questions of surjectivity, periodicity, and contraction in cases a)b): The images of the generators (**??**) show that in these two cases, Φ_f maps H into H. So all iterates Φ_f^n are defined on all of H and given by restrictions of the corresponding iterate acting on $GL_2(\mathbb{Q})$ or $\mathbb{Q}^{2\times2}$. The same relations, or alternatively the eigenvalues, give $\Phi_f^2 = 1$ in case a) and $\Phi_f^4 = 1$ in case b). So $\Phi_f : H \to H$ is an isomorphism, not surjective onto G, and not contracting. It remains to show that no $h \neq 1$ has a lower period. Recalling that we consider matrices up to sign, we are looking for matrices $Q \in H$ with $Q' = \pm Q$ in case a), $Q'' = \pm Q$ in case b). Each equation gives two relations between the four matrix entries, there are only finitely many solutions in $SL_2(\mathbb{Z})$, and the only solution in H is the matrix ± 1.

Questions of surjectivity, periodicity, and contraction in cases c)d): Denoting the range of Φ_f by H', the same arguments as above show that $H' < G$ is a normal subgroup of index 2, generated by S^2, T^2, $V = T^{-1}S^{-1}$, and that a word in $S^{\pm 1}$, $T^{\pm 1}$ belongs to H', if and only if the total numbers of $S^{\pm 1}$ and $T^{\pm 1}$ have the same parity. So Φ_f is an isomorphism $H \to H'$ and not surjective onto G. Since $H' \neq H$, the domain of Φ_f^n will be shrinking with n. Nevertheless there are infinitely many orbits of arbitrary length: for every $h \in G$, h^{2^n} is in the domain of $\Phi_f^{\pm n}$. According to the eigenvalues of Φ_f on $\mathbb{Q}^{2\times 2}$, any n-periodic $Q \in H$ must satisfy $Q' = Q$, which gives $Q = \pm 1$ as a matrix in $SL_2(\mathbb{Z})$ and $Q = 1 \in H$. I do not know if there are other elements with orbit in H defined forever.

The eigenvalues show that Φ_f^{-1} is conjugate to Φ_f in $GL_4(\mathbb{Q})$. A particular conjugation is obtained from the fact that $2A^{-1}$ is conjugate to A in $GL_2(\mathbb{Q})$. It turns out that this inner automorphism of $GL_2(\mathbb{Q})$ restricts to an automorphism χ of G. The automorphism $\chi : G \to G$ defined by $S \mapsto ST$, $T \mapsto T^{-1}$ is an involution. It conjugates the virtual endomorphism $\Phi_f : H \to G$ to the virtual endomorphism $\Phi_f^{-1} : H' \to G$. So if Φ_f was contracting, Φ_f^{-1} would be contracting as well, a contradiction. ∎

When g is a quadratic Thurston map and x_* is not postcritical, the fundamental group $\pi(\widehat{\mathbb{C}} \setminus P,\, x_*)$ acts on the abstract tree of iterated preimages of x_* by monodromy; this action defines the iterated monodromy group. Choosing arcs from x_* to its preimages x_0 and x_1 , it is represented by a contracting wreath recursion and an automaton with finite nucleus [bookN, BN]. Algebraic characterizations of combinatorial equivalence are given in [Kameyama, bookN, algform, BN]. Iterated monodromy groups for quadratic rational Lattès maps f of cases a) and c) are discussed in [pfold, bookN], respectively. They are represented by complex affine maps $z \mapsto \pm z + \lambda$, $\lambda \in \Lambda$, and the virtual endomorphisms divides λ by η

The pure mapping class group G can be described by its action on the fundamental group of $\widehat{\mathbb{C}} \setminus P$. Moreover, for $|P| = 4$, G is identified with the fundamental group of a thrice-punctured sphere [Lodge, KPS, LK]. If a moduli space map k exists, the virtual endomorphism of the iterated monodromy group of k corresponds to the virtual endomorphism of the pure mapping class group G of g. However, there does not seem to be a direct relation between this virtual endomorphism, and the iterated monodromy group of g itself.

5 Twisted Lattès maps of type $(2,\, 2,\, 2,\, 2)$

Analogously to the polynomial case discussed in Section **??**, we may ask to determine the combinatorial equivalence class of a twisted Lattès map $g = \varphi \circ f$:

Proposition 5.1 (Twisted Lattès maps)
Suppose f is a quadratic rational Lattès maps of type $(2,\, 2,\, 2,\, 2)$ with postcritical set P, C is an essential curve in $\widehat{\mathbb{C}} \setminus P$, and the homeomorphism φ represents a

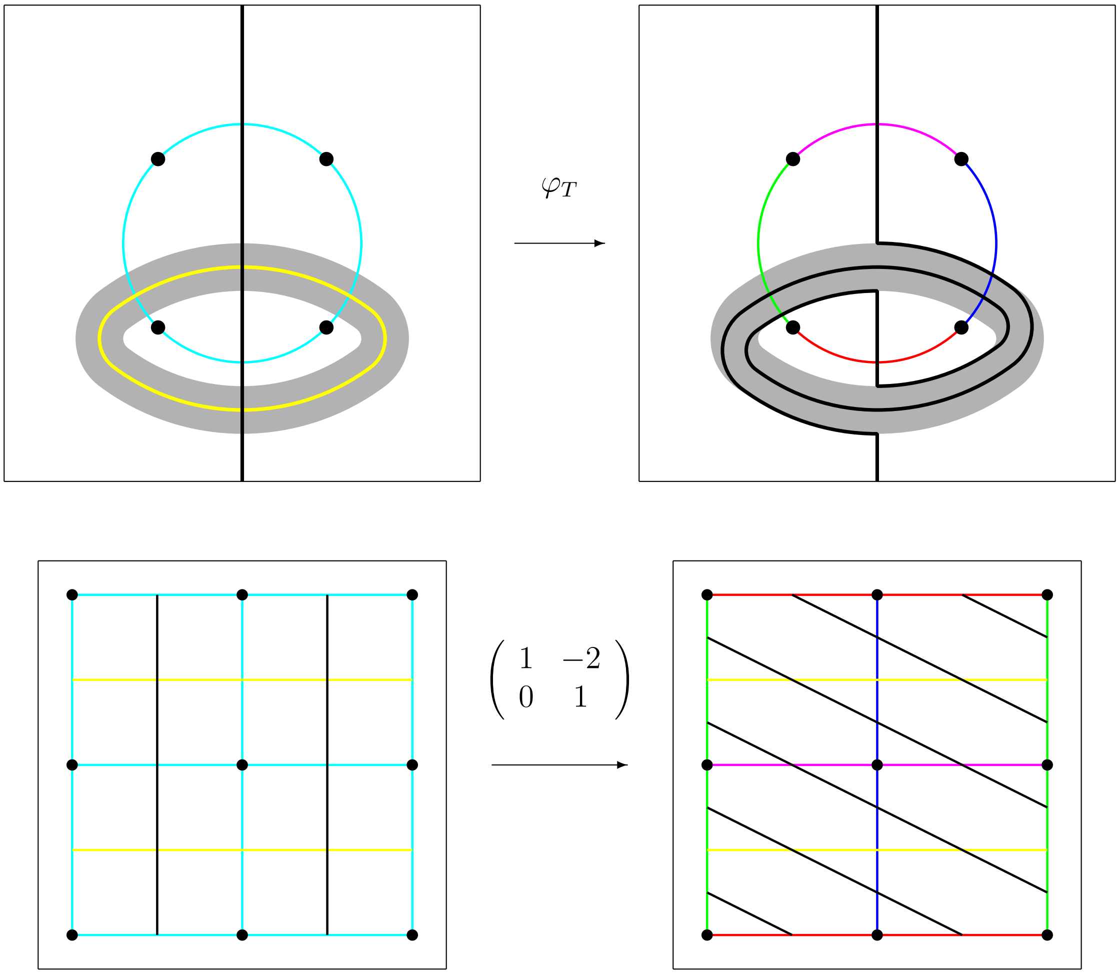

Figure 4: φ_T.

Dehn twist about C. Consider the family of Thurston maps $g_m = \varphi^m \circ f$ for $m \in \mathbb{Z}$. Then:

• *Each combinatorial equivalence class contains at most two maps from the family g_m.*

• *The family g_m provides finitely many maps equivalent to rational functions and infinitely many non-rational classes. Of the latter, at most one is obstructed.*

Proof: By lifting the maps to the torus $\mathbb{R}^2/\mathbb{Z}^2$, f is represented by $\vec{x} \mapsto A\vec{x} + \vec{\kappa}$ and φ by $\vec{x} \mapsto T\vec{x}$. Then $g_m = \varphi^m \circ f$ corresponds to $\vec{x} \mapsto T^m A\vec{x} + \vec{\kappa}$, since T fixes $\vec{\kappa}$ modulo $\mathbb{Z}^2$. By a transformation from $SL_2(\mathbb{Z})$, we may assume that T has the form of (**??**), maybe replacing 2 with any even number. The trace t_m of $T^m A$ is an affine function of m. If t_m was constant, A would have an off-diagonal entry 0, its trace be ± 3 due to determinant 2, and f could not be rational. So for every $t \geq 0$, there are at most two m with $|t_m| = t$. Now according to Sections **??** and **??**, equivalent maps have the same absolute value of the trace $|t_m|$, rational maps are characterized by $|t_m| \leq 2$, and the unique obstructed class of type c) or d) has $|t_m| = 3$. ■

Example 5.2 (Twisted Lattès map)
Consider the Lattès map $f(z) = f_{\mathrm{i}}(z) = \frac{z^2+\mathrm{i}}{1+\mathrm{i}z^2}$ of case b), whose critical values $\pm\mathrm{i}$ are mapped to the common value -1 and then to the fixed point 1. Let $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$

be a simple right Dehn twist about the obvious curve C separating i *and* -1 *from* $-\mathrm{i}$ *and* 1*. and define the Thurston maps* $g_m = \varphi^m \circ f$, $m \in \mathbb{Z}$.

Then for $m > 2$ *and* $m < 0$, g_m *is not equivalent to a rational map and not obstructed. The conjugate map* $f_{-\mathrm{i}}$ *of case b) is obtained for* $m = 2$, *while* $m = 1$ *gives the rational map of case a).*

These statements are an easy application of matrices according to the proof of Proposition **??**, *but we shall discuss numerical and combinatorial arguments in addition, to illustrate the implementation and geometric properties of the Thurston pullback.*

Proof with linear algebra: Choose the torus such that a straight quarter covers a disk with boundary isotopic to the unit circle and $w = 0$ corresponds to $z = 1$. Then f and φ are represented by matrices $\widetilde{A}$ and $\widetilde{T}$ as follows. Note that $\widetilde{A}$ and $\widetilde{T} = \widetilde{T}^1$ can be obtained by applying f or φ to a closed curve according to Remark **??**, lifting curves to $\mathbb{R}^2$, and finding homotopic vectors.

$$\widetilde{T}^m = \begin{pmatrix} 1 & 0 \\ -2m & 1 \end{pmatrix} \quad \widetilde{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \widetilde{T}^m \widetilde{A} = \begin{pmatrix} 1 & 1 \\ -2m-1 & 1-2m \end{pmatrix} \tag{15}$$

For possible comparisons with Theorem **??**, in particular (**??**) and (**??**), let us consider the alternative representations $T = S^{-1}\widetilde{T}S$ and $A = S^{-1}\widetilde{A}S$ as well, with

$$T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} . \tag{16}$$

Either representation gives that the trace is $t_m = 2-2m$, which is even for all $m \in \mathbb{Z}$. According to Proposition **??**.2, this corresponds to the fact that the branch portrait of g_m is of case a)b). Now $t_1 = 0$ shows that g_1 is combinatorially equivalent to the rational map of case a). We have $|t_m| = 2$ for $m = 0$ and $m = 2$, so g_2 is equivalent to f_i or $f_{-\mathrm{i}}$ of case b). Computing η from $T^2 A$ according to (**??**) and (**??**) gives $\eta^2 = +2\mathrm{i}$, so $f_{-\mathrm{i}}$ by the computation in Section **??**. ∎

Note that according to Theorem **??**, all Thurston maps with branch portrait a)b) are obtained by composing f_i with a suitable mapping class, but not all are available from φ^m: We have $|t_m| = 12$ for $m \in \{-5, 7\}$, and $|t_m| = 20$ for $m \in \{-9, 11\}$, but there are at least four different combinatorial equivalence classes in each case according to Example **??**. And recall that the branch portrait is symmetric but the rational map of case a) is not. So this map is equivalent to $\varphi' \circ f_\mathrm{i}$ and to $\varphi'' \circ f_i$, such that the combinatorial equivalence ψ_1 maps $0 \mapsto 0$ or $0 \mapsto \infty$, respectively. Only one of the two possibilities is realized by $g_1 = \varphi^1 \circ f_\mathrm{i}$. I do not know how to see from the matrices, which one it is.

Non-convergent numerical approach: As in Example **??**, we are moving the critical value from $z = \mathrm{i}$ m times counterclockwise around $z = -1$ to define an initial path in moduli space; this corresponds to a continuous deformation from φ^{-m} to the identity in Teichmüller space. Two arguments show that pulling back this path gives a correct implementation of the Thurston Algorithm: first, $g = \varphi^m \circ f_\mathrm{i}$ gives $\sigma_g([\varphi^{-m}]) = [1]$, so we may choose $\psi_0 = \varphi^{-m}$ and $\psi_1 = 1$, analogously to the case of captures and encaptures in Section 6 of [quadJung]. Second, according to Section 2.3 in [quadJung], projecting a path ψ_t connecting ψ_0 to ψ_1 from Teichmüller space to moduli space encodes σ_g. Note that these arguments do not require a moduli space map k.

Using the normalization (**??**) with critical points at 0 and ∞, $u(t) = f_t(0)$ and $-u(t) = f_t(\infty)$, the continuous pullback is given by

$$u(t+1) = \pm\frac{1+u(t)}{1-u(t)} \ , \quad t+1 \ge 2 \tag{17}$$

according to (**??**) in Section **??**. Note that this formula would not be correct for $1 < t+1 < 2$, since in the initial path segment we have an explicit move of $f_t(0) = u(t)$, but $f_t(\infty) = -\mathrm{i} \neq -u(t)$. This gives the alternative pullback relation

$$u(t+1) = \sqrt{\mathrm{i}\,\frac{1+u(t)}{1-u(t)}} \ , \quad 1 < t+1 < 2 \ . \tag{18}$$

The radicand winds m times around 0, so the path $u(t)$ winds $m/2$ times around 0 for $1 \le t \le 2$. In the notation from Figure **??** middle, it follows $\beta\gamma^{-1}$ for $m/2$ rounds; for even m it is a closed path from i to itself, and for odd m it ends at $-\mathrm{i}$. Now this path segment is pulled back indefinitely with (**??**), but it will not converge when g is equivalent to a rational map, because then σ_g has a neutral fixed point. ■

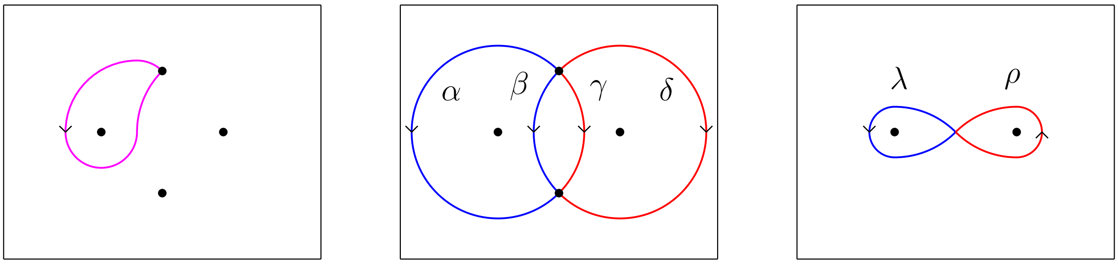

Figure 5: Twisting the Lattès map $f(z) = f_{\mathrm{i}}(z) = \frac{z^2+\mathrm{i}}{1+\mathrm{i}z^2}$ of case b). Left: pushing the critical value $z = \mathrm{i}$ m times counterclockwise around $z = -1$ gives the m-th power of a right Dehn twist about these two points. Middle: according to Example **??**, the curves α, β, γ, δ describe the pullback of the path, when the critical points are marked and the rational maps are even. Right: the curves λ and ρ in the usual moduli space.

Geometric-combinatorial explanation of Example **??**: Although the numerical implementation will not converge, we can interpret the path using two ideas: first, the pullback with (**??**) will be periodic, so the full path is known. Second, g will be equivalent to a rational map, if and only if the path is contractible in moduli space. The latter statement is related to the fact that σ_g is of finite order in the rational case a)b): if σ_g is of finite order, the path in $\mathcal{T}$ will be closed, and since $\mathcal{T}$ is contractible, it projects to a contractible path in $\mathcal{M}$. Conversely, if the periodic path in $\mathcal{M}$ is contractible, it lifts to a closed path in $\mathcal{T}$, so it cannot diverge to the boundary of $\mathcal{T}$, as it must when σ_g has no fixed point. Since we have introduced 0 and ∞ as additional marked points, and $u(t)$ and $-u(t)$ move simultaneously, we should check whether $u(t)$ is contractible in $\widehat{\mathbb{C}} \setminus \{-1, 1, 0, \infty\}$; this claim can be proved by showing that the one-dimensional invariant manifold of the three-dimensional Teichmüller space according to Theorem **??**.3 is contractible. Alternatively, we consider the path in ordinary moduli space: The cross ratio of the four postcritical points defines a new coordinate $w = (u^2+1)/(2u)$, and the projected closed path through $w = 0$ shall be contractible in $\widehat{\mathbb{C}} \setminus \{-1, 1, \infty\}$. The new path will be described in terms of the loops λ and ρ from Figure **??** right. Both α and β^{-1} are mapped to λ, while γ and δ^{-1} correspond to ρ. Path segments are appended from left to right.

When m is even, $u(t)$ is a closed path from i through $-$i back to i for $1 \le t \le 2$. Since $u(2) = u(1) = \mathrm{i}$, the sign + must be used in (**??**) to compute $u(2+0)$ from $u(1+0)$, and by analytic continuation it will be + forever. Now path segments are pulled back as $\alpha \mapsto \beta \mapsto \gamma \mapsto \delta \mapsto \alpha$. The path has period 4 and for $1 \le t \le 5$, $u(t)$ and $w(t)$ follow

$$(\beta\gamma^{-1})^k \cdot (\gamma\delta^{-1})^k \cdot (\delta\alpha^{-1})^k \cdot (\alpha\beta^{-1})^k \ , \qquad (\lambda^{-1}\rho^{-1})^k \cdot (\rho\rho)^k \cdot (\rho^{-1}\lambda^{-1})^k \cdot (\lambda\lambda)^k \tag{19}$$

when $m = 2k > 0$, while for $m = -2k < 0$ we have

$$(\gamma\beta^{-1})^k \cdot (\delta\gamma^{-1})^k \cdot (\alpha\delta^{-1})^k \cdot (\beta\alpha^{-1})^k \ , \qquad (\rho\lambda)^k \cdot (\rho^{-1}\rho^{-1})^k \cdot (\lambda\rho)^k \cdot (\lambda^{-1}\lambda^{-1})^k \ . \tag{20}$$

Since the fundamental group $\pi(\widehat{\mathbb{C}} \setminus \{-1,\, 1,\, \infty\},\, 0)$ is free on the generators λ and ρ, we see that the path is contractible only when $m = 2$. Then a path segment of length 2 is not contractible, so g_2 defines a Thurston pullback map of finite order 4, and g_2 is equivalent to a rational map of case b), f_i or $f_{-\mathrm{i}}$.

When m is odd, $u(1) = \mathrm{i}$ and $u(2) = -\mathrm{i}$ shows that the sign $-$ must be used in (**??**). Path segments are pulled back according to $\alpha \leftrightarrow \gamma^{-1}$, $\beta \leftrightarrow \beta^{-1}$, and $\delta \leftrightarrow \delta^{-1}$. The path has period 2 and for $1 \le t \le 3$, $u(t)$ and $w(t)$ are described as

$$(\beta\gamma^{-1})^k\beta \cdot (\beta^{-1}\alpha)^k\beta^{-1} \ , \qquad (\lambda^{-1}\rho^{-1})^k\lambda^{-1} \cdot (\lambda\lambda)^k\lambda \tag{21}$$

when $m = 2k+1 > 0$, while for $m = -(2k+1) < 0$ we have

$$(\gamma\beta^{-1})^k\gamma \cdot (\alpha^{-1}\beta)^k\alpha^{-1} \ , \qquad (\rho\lambda)^k\rho \cdot (\lambda^{-1}\lambda^{-1})^k\lambda^{-1} \ . \tag{22}$$

Now the path is contractible only for $m = 1$, g_1 gives a Thurston pullback map of finite order 2, and g_1 is equivalent to the rational map of case a). — This approach shows that $g_m = \varphi^m \circ f_\mathrm{i}$ is not equivalent to a rational map when $m < 0$ or $m > 2$, but I have not tried to determine equivalence classes from the path. Note that two different moduli spaces were used: $u(t)$ is obtained easily from the simple pullback relation (**??**), while $w(t)$ is easier to check for contractibility. ∎

Although the previous arguments are rather unique to Thurston maps of case $(2,\, 2,\, 2,\, 2)$ a)b), they provide an explicit example of a pullback in moduli space, which is neither convergent nor divergent to the boundary. When f is case c) or d) of type $(2,\, 2,\, 2,\, 2)$, the combinatorial equivalence class is obtained analogously from matrices, e.g., with $t_m = 1 - 2m$. The pullback of curves will no longer be periodic: for $t_m = -1$ it performs an irrational rotation in a bounded subset of $\mathcal{M}$. For $|t_m| = 3$ it diverges to the boundary of $\mathcal{M}$, while for $|t_m| > 3$, it is bounded in $\mathcal{M}$ but lifts to an unbounded path in $\mathcal{T}$. I have not tried to obtain the path combinatorially, but in principle the lift from $\mathcal{M} = \widehat{\mathbb{C}} \setminus \{0,\, 1,\, \infty\}$ to the upper halfplane $\mathcal{T}$ can be obtained as follows: draw a triangle between the punctures of $\mathcal{M}$ and mark the corresponding fundamental domains of the modular function in the upper halfplane. Then lift the path by recording it crossing the triangle edges. For $|t_m| = 3$ the path in $\mathcal{M}$ will converge to a puncture and the lifted path will converge to a rational boundary point. For $|t_m| > 3$ the path in $\mathcal{T}$ shall converge to an irrational boundary point.

Remark 5.3 (Behavior of the pullback)
*1. In Example **??** we have used the "even" covering space according to Remark **??**.3 to obtain the curves more easily. Independently of this, for g with the branch portrait a)b), σ_g is of finite order, if and only if g is equivalent to a rational map. For $\varphi^m \circ f_i$ with m even, the pullback of $\psi_1 = 1$ is such that $\pi([\psi_n])$ is constant; additional information is required to see whether the map is equivalent to $f_{-\mathrm{i}}$ or not rational.*

*2. When twisting the Rabbit polynomial $\varphi^m \circ f_R$, $\pi([\psi_n])$ will be constant for a finite number of steps and then converge. For the particular Dehn twist φ according to Example **??**, it does not seem to be constant forever; even if g_m is equivalent to f_R, the path jumps away and converges back. For twists about different curves, it does happen that a segment is pulled back to a trivial path, and then it stays trivial.*

*3. These observations have a partial explanation in terms of the virtual endomorphism Φ_f of the pure mapping class group G: for $h \in H < G$ we have $h \circ f = f \circ h'$ and $\Phi_f(h) = h'$. Now the Thurston pullback for $\varphi^m \circ f$ with $\psi_0 = \varphi^{-m}$ and $\psi_1 = 1$ is obtained recursively as $\psi_{n+1} = \Phi_f(\psi_n \circ \varphi^m)$, as long as this is defined. There may be a minimal n with $\psi_n \circ \varphi^m \notin H$; then we still have $f_n = f$ in our normalization of critical points, but $\pi([\psi_{n+1}]) \neq \pi([\psi_n]) = \pi([1])$. Now ψ_{n+1} is no longer described in terms of G. The interpretation of finite and infinite orbits in G depends on whether we are in case a)b), case c)d), or not of type (2, 2, 2, 2). In Example **??** we have the following virtual endomorphism, described in terms of the same matrices S and T as in (**??**), but acting differently from (**??**) since here $-A = A_{-2}$:*

$$\Phi_f \ : \quad S \mapsto T^2 \mapsto TST^{-1} \mapsto S^{-1}T^{-1}S^{-1}T^{-1} \mapsto S \tag{23}$$

For $m = 2$ now $\psi_{n+1} = \Phi_f(\psi_n \circ T^2)$ gives

$$\psi_0 = T^{-2} \qquad \psi_1 = 1 \qquad \psi_2 = TST^{-1} \qquad \psi_3 = S^{-1}T^{-2} \qquad \psi_4 = T^{-2} = \psi_0 \ , \tag{24}$$

so $g = \varphi^2 \circ f_{\mathrm{i}}$ satisfies $\sigma_g^4([\psi_0]) = [\psi_0]$ and is equivalent to a rational map of case b).