Small connectedness loci in parameter spaces

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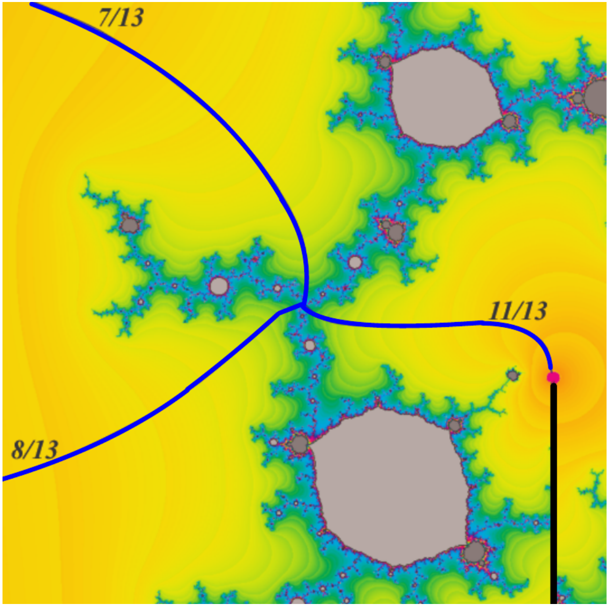
The images are made with Mandel, a program available from www.mndynamics.com .

Talk given at Workshop on Holomorphic Dynamics, Holbæk, October 21, 2023.

1a **Motivation from a picture of S_4, courtesy Jack Milnor**

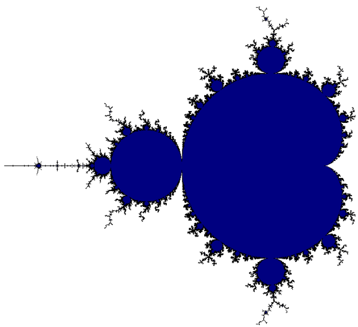
These cubic polynomials $f(z)$ have a persistently 4-periodic critical point. Various hyperbolic components can be seen in the non-escape locus. Can we infer their type from the shape?

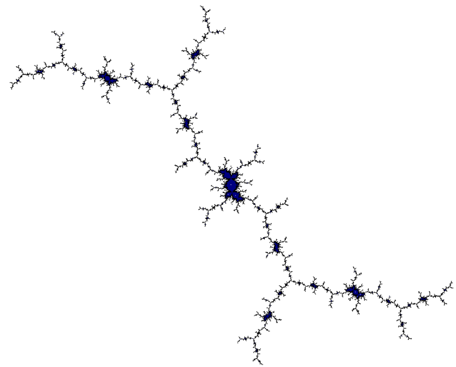


A decomposition of the Julia sets, for example by renormalization, is useful both for understanding the dynamics and topology of filled Julia sets $\mathcal{K}$, and for the structure of parameter spaces.

1b **Recall quadratic polynomials**

For $c \in \mathcal{M}$, the critical orbit of $f_c(z) = z^2 + c$ is bounded and the filled Julia set $\mathcal{K}_c$ is connected. The images show the Mandelbrot set $\mathcal{M}$ and an example of $\mathcal{K}_c$, such that $z = 0$ is 12-periodic under f_c.

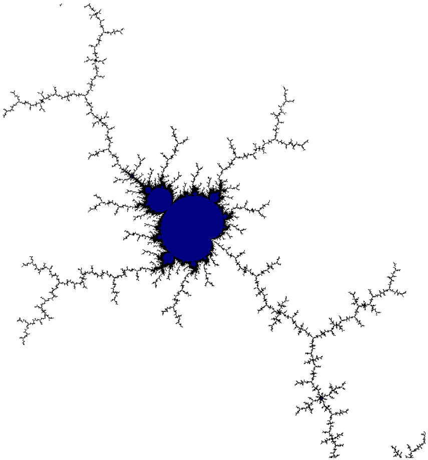


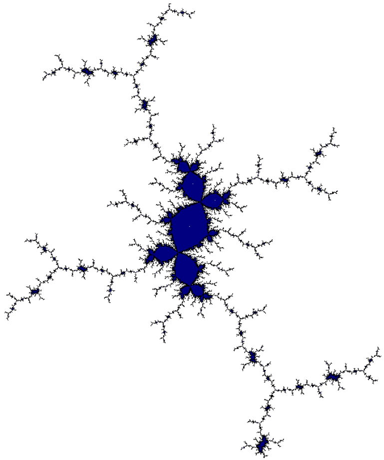


1c **Renormalization and straightening**

The renormalization $g_c(z) = f_c^4(z) : U'_c \to U_c$ is quadratic-like and can be straightened to another quadratic polynomial $f_{\widehat{c}}$.

$\mathcal{M}$ contains a corresponding copy $\mathcal{M}_4$ of itself, with infinitely many decorations attached.





1d **Straightening of quadratic-like families**

Theorems by Adrien Douady and John Hamal Hubbard:

1) Suppose that $g_c(z) = f_c^n(z) \; : \; U_c' \to U_c$ is quadratic-like, the quasi-disks move holomorhically for $c \in U_{\scriptscriptstyle M}$, and $v_c - \omega_c$ winds once around 0.
Then there is a straightening map $\chi : \mathcal{M}_n \to \mathcal{M}$, $c \mapsto \widehat{c}$. Its inverse is the tuning map $\widehat{c} \mapsto c = c_n * \widehat{c}$.

In the satellite case, sublimbs may be treated separately.

2) For every hyperbolic component $\Omega_n \subset \mathcal{M}$, there is a corresponding small Mandelbrot set $\mathcal{M}_n$, and their union is the locus of simple renormalization.

Note that to apply Theorem 1, the required domains may be constructed, for example, as explicit disks, from asymptotic formulas, or as puzzle-pieces.

2a **Two-parameter families of rational maps**

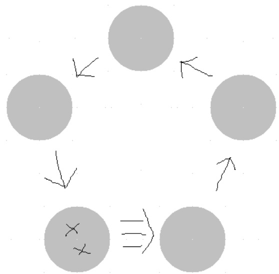
While quadratic polynomials form a one-parameter family, natural two-parameter families, which have two free critical points, are as follows:

Cubic polynomials have a superattracting fixed point at ∞ and two finite critical points. Up to affine conjugation they may be parametrized, for example, as $f(z) = z^3 - 3a^2 z + b$ or $f(z) = A(z^3 - 3z) + B$.

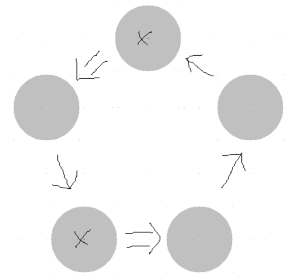
Quadratic rational maps have two critical points. Up to Möbius conjugation they may be parametrized, for example, as $f(z) = \frac{z^2+A}{z^2+B}$ or $f(z) = c(z+\frac{1}{z})+d$.

2b **Types of hyperbolic maps and components**

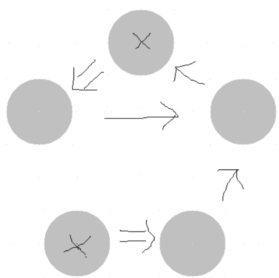
The Fatou components of a hyperbolic map with connected Julia set may be mapped as follows; this is used to classify hyperbolic components as well:



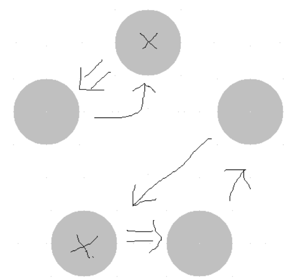
A – adjacent



B – bitransitive



C – capture



D – disjoint

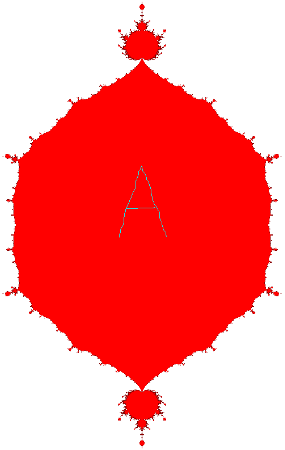
2c **One-parameter families as one-dimensional slices**

A critical relation like $f^3(\omega_1) = f^7(\omega_2)$ defines a one-parameter subfamily of cubic polynomials or quadratic rational maps, respectively. Here consider maps with an n-periodic critical point:

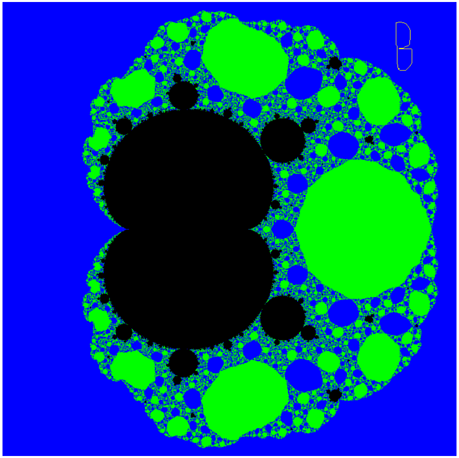
$S_n = \{f(z) = z^3 - 3a^2z + b \,|\, z = -a \text{ is } n\text{-periodic}\}\,,$

$V_n = \{f(z) = \frac{z^2+A}{z^2+B} \,|\, z = \infty \text{ is } n\text{-periodic}\}\,.$

S_1



V_2



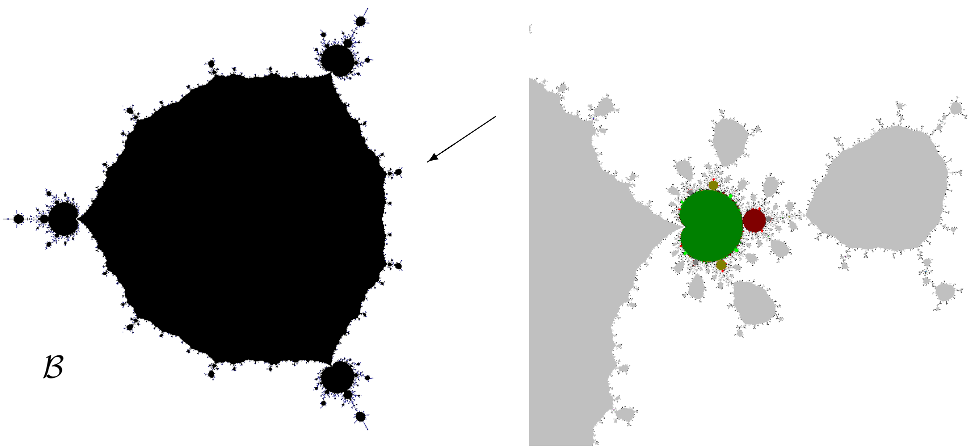
2d **Two families of even quartic polynomials**

Any even quartic polynomial is a composition of two quadratic polynomials; interchanging these gives a semiconjugate quartic polynomial. Here consider quartic maps with a fixed critical point:

$(Q \circ P)(z) = (z^2 - q^2)^2 + q$, and

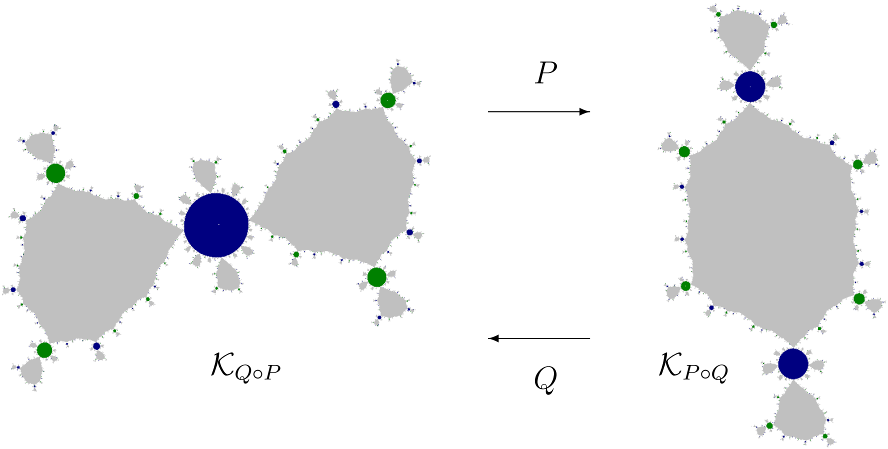
$(P \circ Q)(z) = (z^2 + q)^2 - q^2$.

The common connectedness locus is denoted by $\mathcal{B}$.



2e **A model using two disjoint planes**

The dynamics and combinatorics of $(Q \circ P)(z) = (z^2 - q^2)^2 + q$ and $(P \circ Q)(z) = (z^2 + q)^2 - q^2$ may be described by considering $P(z) = z^2 - q^2$ and $Q(z) = z^2 + q$ as maps between two different planes; compare the mapping sheme according to Jack Milnor:



Copies of $\mathcal{B}$ and of these Julia sets will be seen in the following examples; since anti-matings may be conjugated with a rotation, V_2 shows $(\mathcal{B})^3$ instead.

3a **Straightening around a hyperbolic component**

What small set is expected around a hyperbolic component?

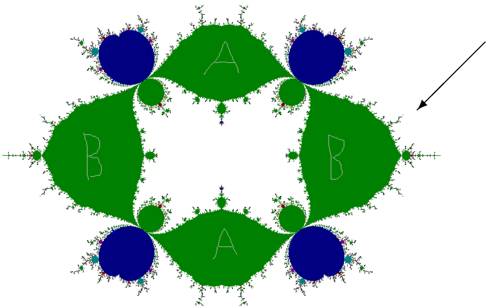
Type	two-parameter family	S_n or V_n
Adjacent	cubic connectedness locus	S_1
Bitransitive	even quartic polynomials	$\mathcal{B}$
Capture	$\{(c,\, z) \,\|\, c \in \mathcal{M},\; z \in \mathcal{K}_c\}$	$\sim$ disk
Disjoint	$\mathcal{M} \times \mathcal{M}$	$\mathcal{M}$

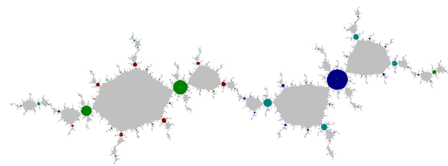
The two-dimensional straightening map of cubic polynomials was discussed by Hiroyuki Inou and Jan Kiwi; it is injective and it may be discontinuous at parabolic parameters, but the one-dimensional map shall be continuous.

Now if f in S_n or V_n has a Fatou component of type A or B, the renormalization $f_c^n : U' \to U$ has a fixed critical point, so its straightening will be in S_1 or $\mathcal{B}$, respectively. It remains to show that the renormalization locus gives all of S_1 or $\mathcal{B}$.

3b **Example:** S_2

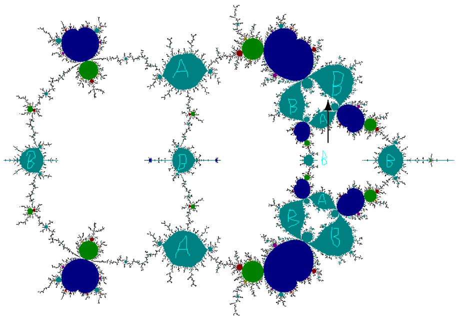
The image of S_2 shows two hyperbolic components with adjacent dynamics and two with bitransitive dynamics; actually the parametrization is a two-fold cover of moduli space.

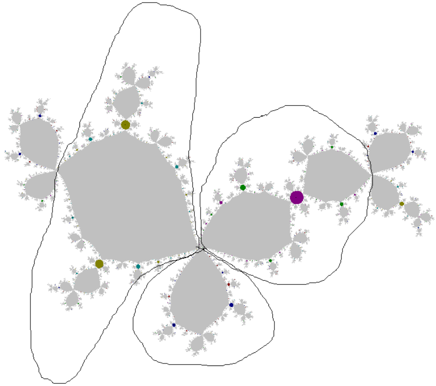




3c **Example:** S_3

In S_3 there are four hyperbolic components with adjacent dynamics and eight of bitransitive type.

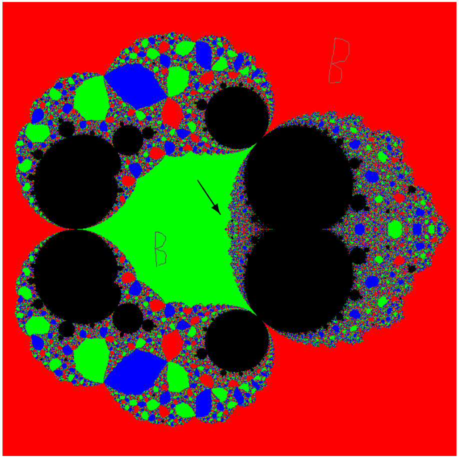


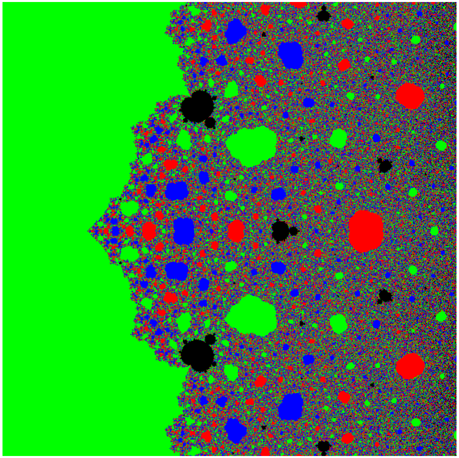


By the way, here is a zoom video around a Misiurewicz point: [misiS3].

3d **Eample:** V_3

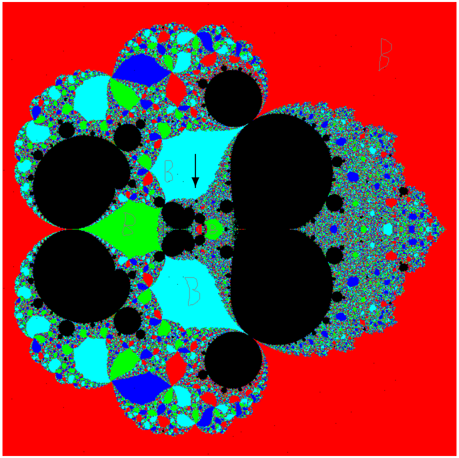
Quadratic rational maps in V_n do not have hyperbolic components of adjacent or escaping type. In V_3 there are two components with bitransitive dynamics. (The Julia sets and dynamics will be discussed later on p. 6b.)

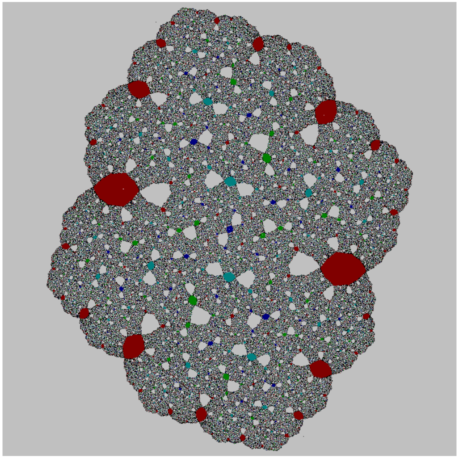




3e **Eample:** V_4

In V_4 there are six bitransitive hyperbolic components.

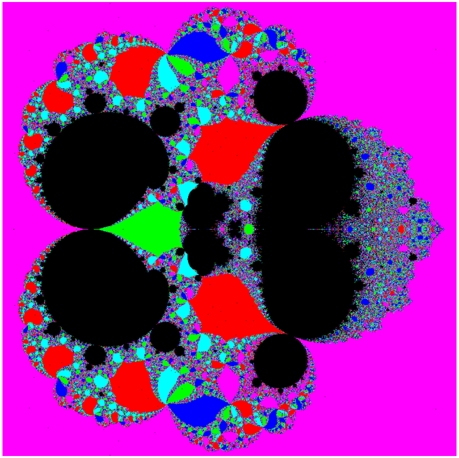


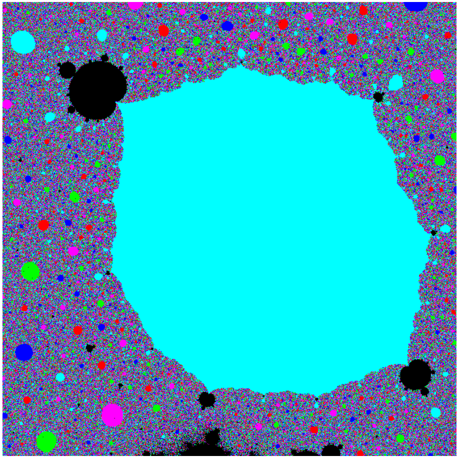


On the right hand side, copies of $\mathcal{K}_{Q\circ P}$ contain green or blue disks, while the unique copy of $\mathcal{K}_{P\circ Q}$ around ∞ has red disks.

3f **Example:** V_5

This parametrization of V_5 contains the square-root of a quartic polynomial, so there are two sheets of parameter space; both contain bitransitve components, and some are primitive.





4a **Conjectural description**

For any bitransitive hyperbolic component Ω in S_n or V_n , I expect a surrounding copy of $\mathcal{B}$ in parameter space.

For V_n , a big limb may be missing, and in V_2 we have $(\mathcal{B})^3$.

In the dynamic plane, we can find copies of $\mathcal{K}_{Q\circ P}$ and $\mathcal{K}_{P\circ Q}$ replacing Fatou components.

What does that mean and how can we prove it?

4b **Approach with renormalization**

Analogously to p. 1d, we may try to construct an analytic family of quartic-like restrictions of the first-return map, or a pair of quadratic maps.

This works well in certain cases, but in general there will be two problems:

The small Julia sets may have common boundary points; then the disks $U_c' \to U_c$ may be hard to define or to control. In the case of V_n, there may be infinitely many common boundary points in fact.

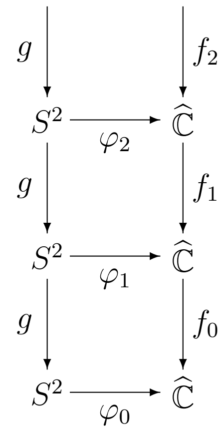
Even in S_n, in the primitive case, we need to know more about landing properties of external rays, to define define these disks as puzzle-pieces.

4c **Approach with Thurston theory**

In the postcritically finite case, we may construct (or decompose) a map as follows:

Define a branched covering $g : S^2 \to S^2$, which need not be holomorphic; it shall have the desired combinatorics of critical orbits.

Choose a homeomorphism $\varphi_0 : S^2 \to \widehat{\mathbb{C}}$ and pull it back with a sequence of rational maps f_n according to the commuting diagram.



Under suitable assumptions, the rational maps converge to the desired one, and the homeomorphisms converge up to homotopy (relative to the marked points).

Actually, the pullback map is defined on a Teichmüller space.

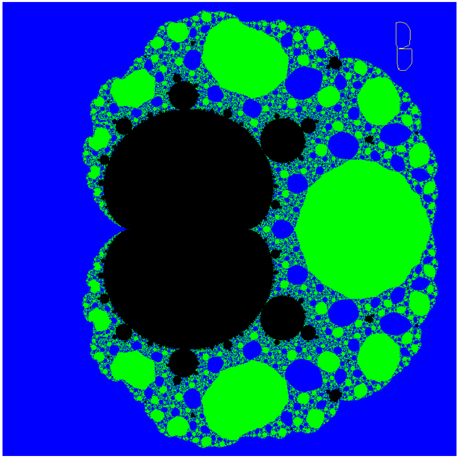
4d **Comparison**

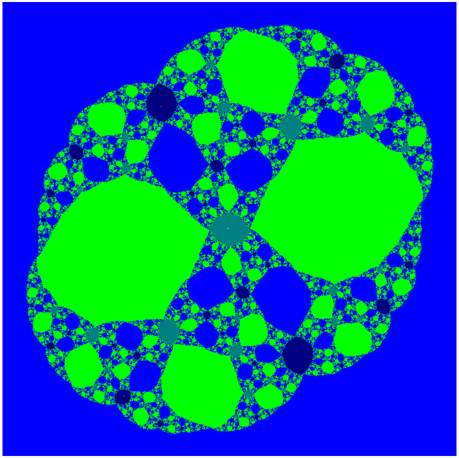
Renormalization and straigthening	Thurston theory
may be postcritically infinite	mostly postcritically finite
continuous dependence on parameters	continuity is not proved
straightening by quasi-conformal surgery, tuning is defined as the inverse of straightening	both directions are possible
fairly easy theory, no numerics so far, but combinatorial description	simple numerical implementation
Fatou components shall touch only finitely often	Fatou components may touch infinitely often

5a **Anti-mating**

Anti-mating or cross-mating $f \simeq P \prod Q$ is a special case of the previous discussions, starting with the bitransitive component around $f(z) = 1/z^2$. And it can be defined analogously to mating, starting with a formal anti-mating $P \sqcap Q$ and collapsing postcritical ray-equivalence classes.

When $p = -q^2$, the resulting map is in V_2. See a phase space video [mateK-Banti] and a deformation video [slowanti2]. Note that the big limb of $(\mathcal{B})^3$ is missing. For $\partial\mathcal{B}$, this construction goes back to Vladlen Timorin.





5b **Properties of anti-matings**

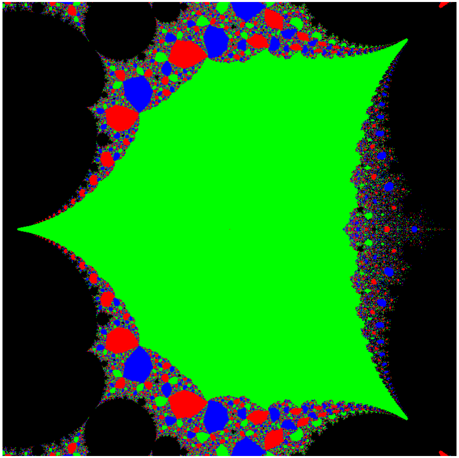
Using a definition in terms of laminations, Davoud Ahmadi Dastjerdi has shown that a formal anti-mating $P \sqcap Q$ has only removable obstructions, whenever the three fixed rays land at different fixed points. So there is a corresponding rational map f, which is a combinatorial anti-mating and a geometric anti-mating, with a possible exception for type $(2,\,2,\,2,\,2)$.

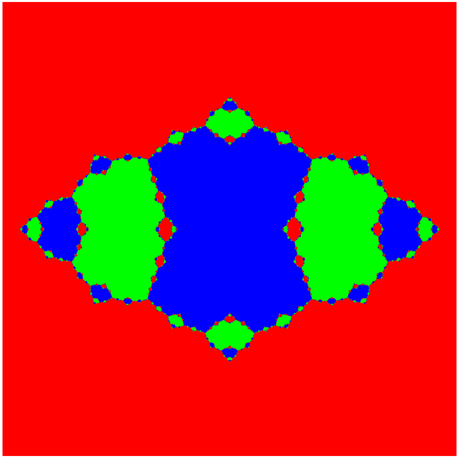
We also have the mating $f^2 \simeq (Q \circ P) \coprod (P \circ Q)$, so why do we need anti-matings? Staying in the quadratic family may simplify understanding the combinatorics and dynamics, and we may use the Levy cycle criterion. Anti-matings without postcritical identifications, especially hyperbolic anti-matings, are characterized by an anti-equator; this was shown independently by Ma Liangang.

Since the two Julia sets share all of their boundaries, anti-matings cannot be constructed by renormalization, and only geometrically finite maps can be treated directly with Thurston theory.

6a **Generalizations of anti-mating**

Starting from the 3-periodic bitransitive map $f(z) = 1 - \frac{1}{z^2}$ in V_3, there is a construction analogous to anti-mating. The relevant Fatou components share infinitely many points. Probably the only non-removable obstruction happens when 1/3 and 2/3 land together, so $\mathcal{B}$ loses only one of the three big limbs.

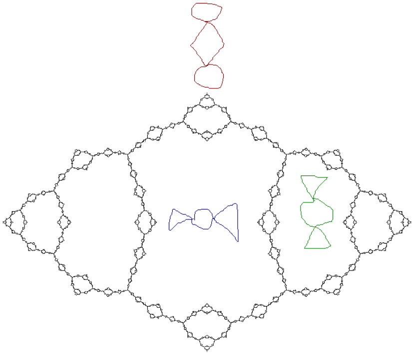


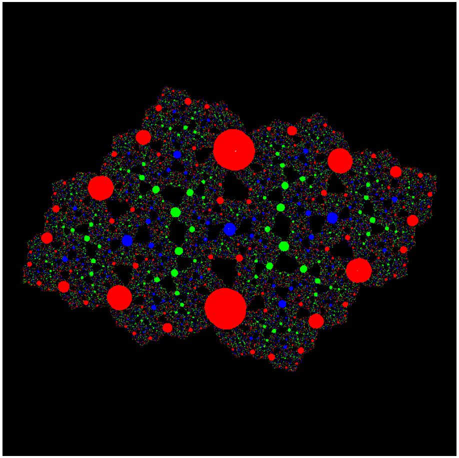


6b **Bitransitive components in V_3**

For the example of V_3 from page 3d, the sketch shows how $\mathcal{K}_{P\circ Q}$ and $\mathcal{K}_{Q\circ P}$ shall appear within the three periodic components.

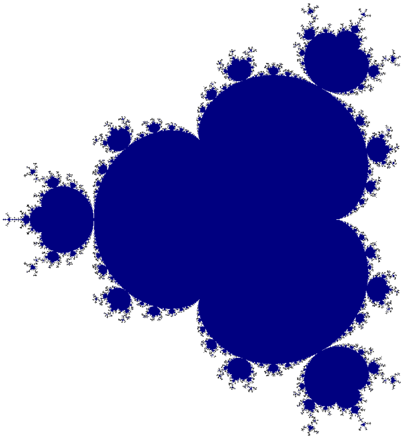
[A deformation video is under construction.]

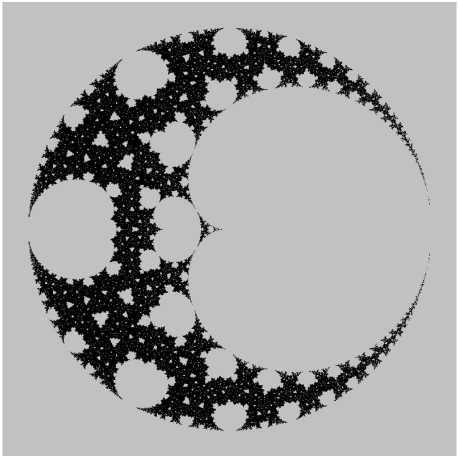




6c **Small Multibrot sets in persistently bitransitive families**

When anti-mating is performed with $P(z) = z^2$ and $Q(z) = z^2 + q$, the relevant polynomial connectedness locus is the Multibrot set $\mathcal{M}_4$ for $(Q \circ P)(z) = z^4 + q$ and $(P \circ Q)(z) = (z^2 + q)^2$. The resulting rational map is of the form $f_a(z) = 1 + \frac{a}{z^2}$ with $f_a(0) = \infty$, and we see a copy of $(\mathcal{M}_4)^3$ around the outer component of period 2. See also the approach of Pascale Roesch and Bastien Rossetti, which uses the mating $f_a^2 \simeq (Q \circ P) \coprod (P \circ Q)$ and a continuation to the molecule along satellite components. A deformation video [slowanti0].



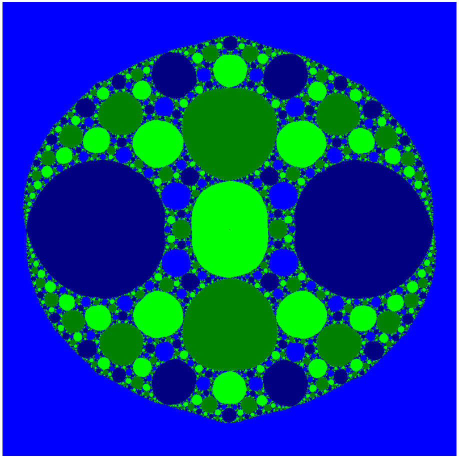


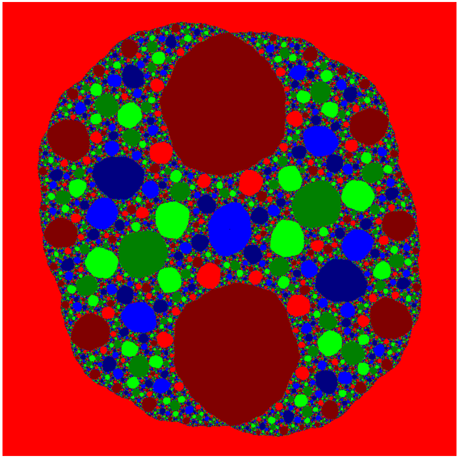
6d **Anti-mating for a persistently bitransitive family**

Analogously to the discussion of V_3, other parts of parameter space can be described by starting from the 3-periodic $f(z) = 1 - \frac{1}{z^2}$.

The images show the Julia sets for $(z^2)\prod(z^2-1)$ of period 4 and the generalized anti-mating of period 6.

[A deformation video is under construction.]



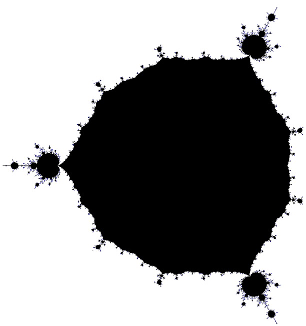
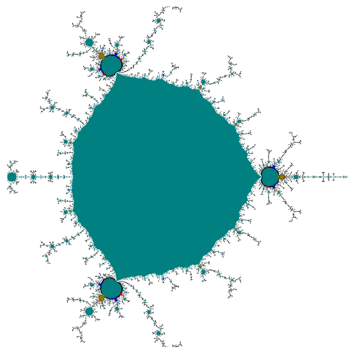
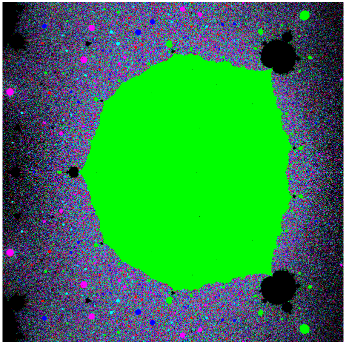


7 Conclusion

S_n and V_n seem to contain small copies of $\mathcal{B}$, and also S_1 in the former case. These are recognizable from their shapes.

Partial proofs may be based on renormalization or on Thurston theory. A complete proof for all n will require a deeper understanding of the combinatorics, for example landing properties of external rays or realizability of laminations.

The images show $\mathcal{B}$ again, and subsets of S_3 and V_5 .

Hjertelig tak !