Local and asymptotic similarity in one-parameter families.

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One-parameter families of complex polynomials or rational mappings show various phenomena of self-similarity in the parameter space and of similarity to the Julia sets.

1. Phenomena at Misiurewicz points of quadratic polynomials.
2. Cubic polynomials with marked critical points.
3. Non-degenerate Misiurewicz bifurcation.
4. Degenerate and parabolic examples.
5. Rational families.

The images were made with Mandel, a program available from www.mndynamics.com .

Talk given at the NextStep workshop in Holbæk, October 2009.

1. Phenomena at Misiurewicz points of quadratic polynomials:

Quadratic polynomial $f_c(z) = z^2 + c$,
filled Julia set $\mathcal{K}_c = \{z \,|\, f_c^n(z) \not\to \infty\}$,
Mandelbrot set $\mathcal{M} = \{c \,|\, c \in \mathcal{K}_c\} = \{c \,|\, f_c^n(0) \not\to \infty\}$.
$\mathcal{M}$ is the non-escape locus and the connectedness locus of quadratic polynomials. The bifurcation locus is $\partial\mathcal{M}$.

A Misiurewicz point is a parameter $c = a$, such that the critical point $z = 0$ is strictly preperiodic. The corresponding periodic cycle is always repelling. The multiplier ρ_a appears as a scaling factor of $\mathcal{M}$ at a.

The examples deal with the α-type Misiurewicz point of preperiod 3 in the $1/3$-limb.

Example 1.1: asymptotic similarity.

Example 1.2: asymptotic models on multiple scales.

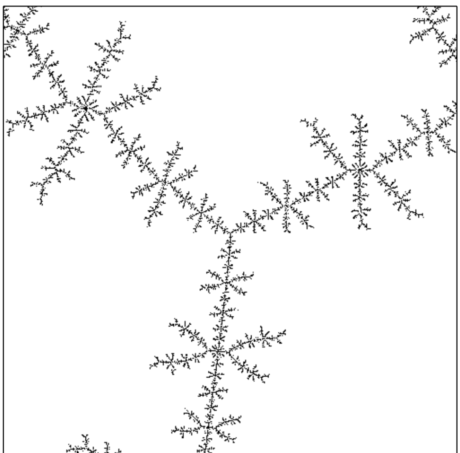
Example 1.3: local similarity.

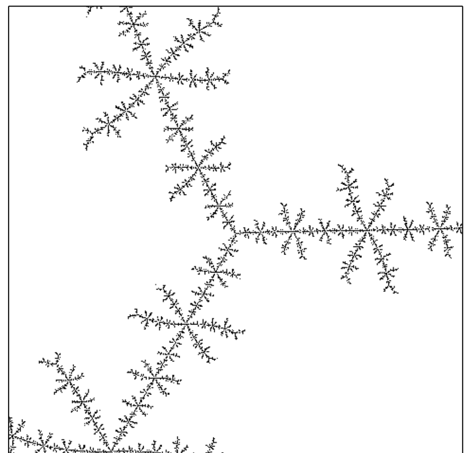
Example 1.1: Asymptotic similarity

There is a sequence of centers c_n and little Mandelbrot sets of period $n+4$, such that $c_n \sim a + K\rho_a^{-n}$, and the diameter of the little Mandelbrot sets is $\sim |\rho_a|^{-2n}$.

Define the Koenigs conjugation $\phi_c(z)$, the asymptotic model set X_c, which is ρ_c-invariant and locally given by $\phi_c(\mathcal{K}_c)$, and $u(c) = \phi_c(f_c^3(c))$.

In Hausdorff-Chabauty distance, $\rho_a^n(\mathcal{M} - a) \to X_a/u'(a)$ [TL].



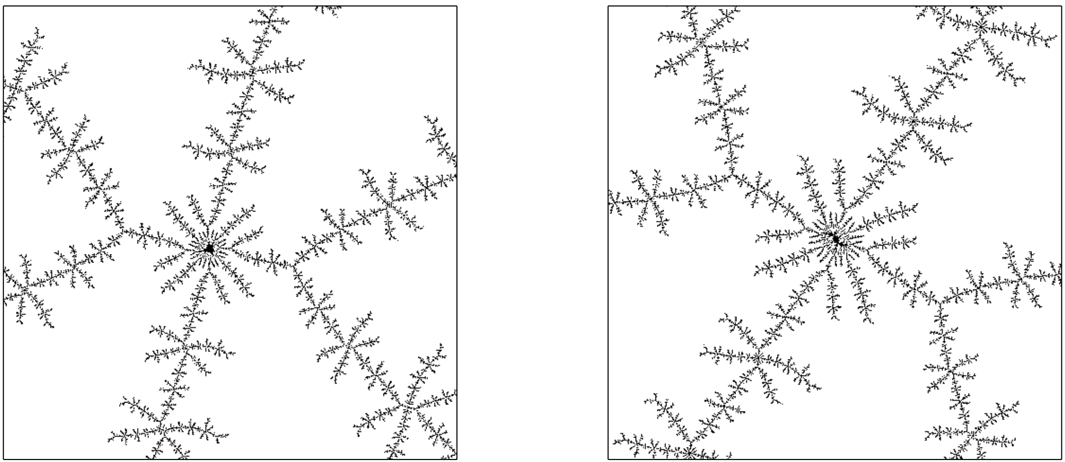


Example 1.2: Asymptotic similarity on the next level.

In Hausdorff-Chabauty distance,

$$\rho_a^n\left(\mathcal{M} - c_n\right) \to A_0\left(X_a - \phi_a(0)\right),$$
$$\rho_a^{3/2\,n}\left(\mathcal{M} - c_n\right) \to A_1\left(X_a - \phi_a(0)\right)^{1/2},$$
$$\rho_a^{7/4\,n}\left(\mathcal{M} - c_n\right) \to A_2\left(X_a - \phi_a(0)\right)^{1/4} \ \ldots$$

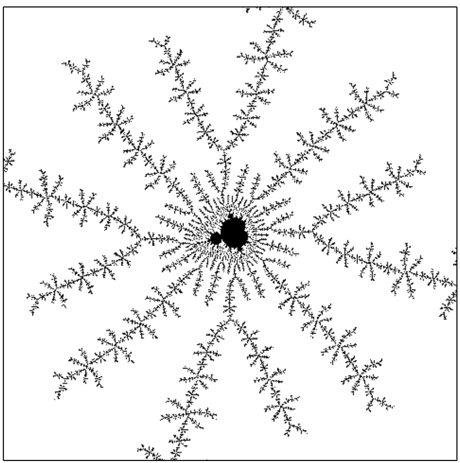
A little Mandelbrot set of period 16 is centered at c_{12}. Observe the asymptotic model of level 1.

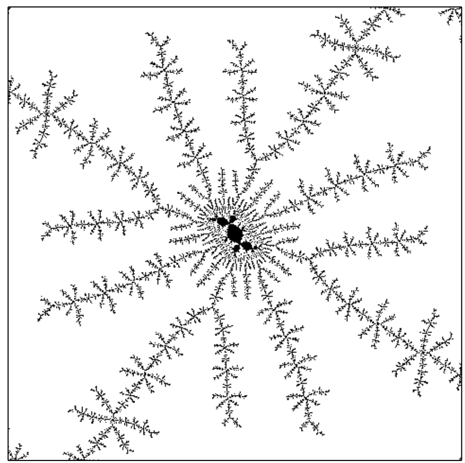


Example 1.3: Local similarity between decorations of the little Mandelbrot set and a little Julia set.

Rescale both planes by ρ_a^{2n}. Then $f_c^{n+4}(z)$ is close to $\overline{z}^2 + \overline{c}$, since the higher powers of $\overline{z}$ contain negative powers of ρ_a. A straightening close to the identity gives $\hat{z}^2 + \hat{c}$.

The inner parts of the dynamic decorations are iterated to a set far out, which is close to an asymptotic model. The k-th iterate is approximately $(\Phi_{\hat{c}}(\hat{z}))^{2^k}$, where $\Phi_{\hat{c}}(\hat{z})$ denotes the Boettcher conjugation. Thus $\Phi_{\hat{c}}(\hat{z})$ maps decorations to a set that is slowly varying with the parameter. Therefore parameter decorations are mapped to the same set by $\Phi_M(\hat{c})$.





2. Cubic polynomials with marked critical points:

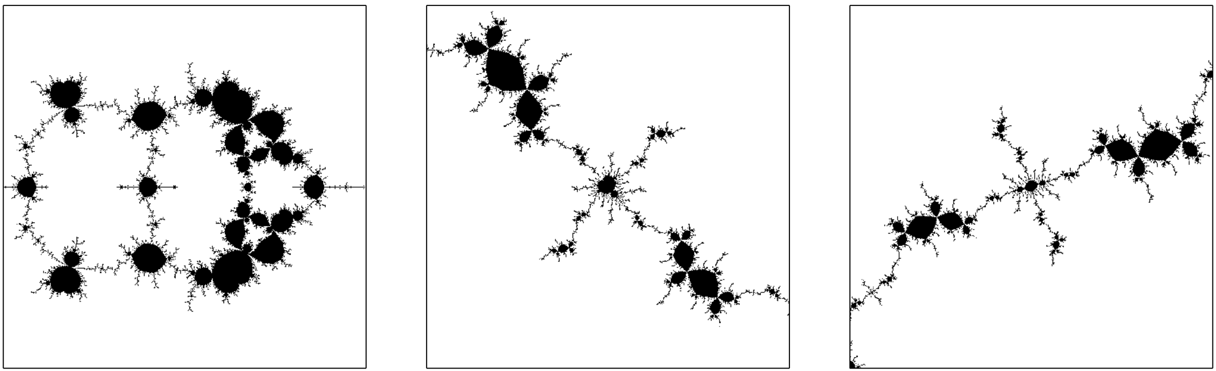
One-parameter family of polynomials $f_c(z)$, c in domain or one-dimensional manifold $\mathbb{C}_*$. Suppose that (locally) the critical points are *marked*: they are analytic functions $c \mapsto \omega_c^j$ of the parameter c. Consider
non-escape loci $\mathcal{M}^j = \{c \,|\, \omega_c^j \in \mathcal{K}_c\} = \{c \,|\, f_c^n(\omega_c^j) \not\to \infty\}$,
the connectedness locus $\mathcal{C} = \bigcap \mathcal{M}^j$, and
the bifurcation locus $\mathcal{B} = \bigcup \partial\mathcal{M}^j$.

For cubic polynomials $f_c(z) = A_c(z^3 - 3z) + b_c$, the critical points are $\omega_\pm = \pm 1$.
Non-escape loci $\mathcal{M}_\pm = \{c \,|\, \pm 1 \in \mathcal{K}_c\} = \{c \,|\, f_c^n(\pm 1) \not\to \infty\}$,
the connectedness locus $\mathcal{C} = \mathcal{M}_- \cap \mathcal{M}_+$, and
the bifurcation locus $\mathcal{B} = \partial\mathcal{M}_- \cup \partial\mathcal{M}_+$.

Example 2.1: persistently attracting cycle, $M_- = \mathbb{C}_*$.
Example 2.2: critical relation, $\mathcal{M}_- = \mathcal{M}_+$.
Example 2.3: persistently neutral cycle, $\mathcal{M}_- \cup \mathcal{M}_+ = \mathbb{C}_*$.
Example 2.4: arbitrary slice, $\mathcal{M}_-$ and $\mathcal{M}_+$ are independent.

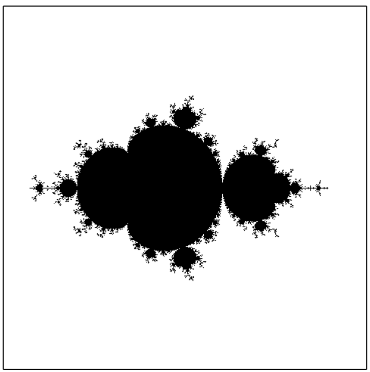
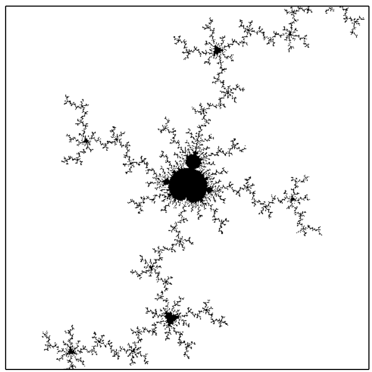
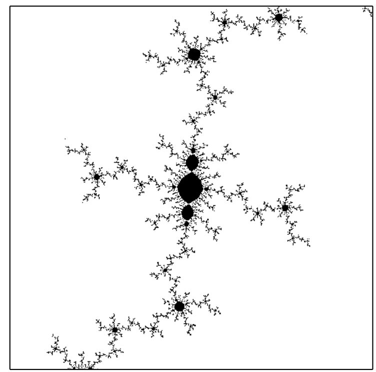
Example 2.1: $f_c(z)$ has a persistently superattracting 3-cycle.
Thus $\mathcal{M}_- = \mathbb{C}_*$, $\mathcal{C} = \mathcal{M}_+$, and $\mathcal{B} = \partial\mathcal{M}_+$.

The middle image shows a decorated little Mandelbrot set in $\mathcal{C} = \mathcal{M}_+$ on a vein to a Misiurewicz point. The right image shows the corresponding subset of $\mathcal{K}_c$ to illustrate local similarity.



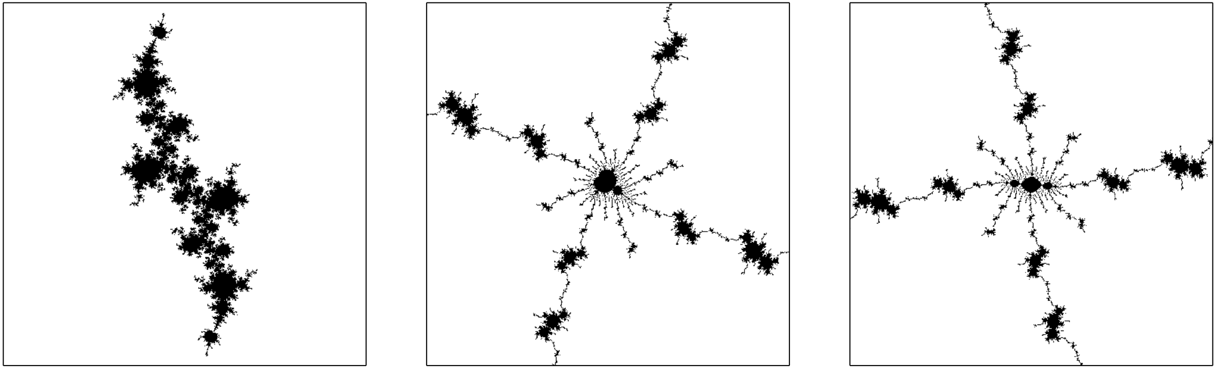
Example 2.2: The family $f_c(z) = c(z^3 - 3z) + (1 - 2c)$ satisfies $-1 \mapsto 1$ for all parameters c. Thus the critical orbits coincide, and $\mathcal{M}_- = \mathcal{M}_+$.

Hyperbolic components of capture type are related to little quadratic Mandelbrot sets, and hyperbolic components of bitransitive type are related to quartic Multibrot sets. The middle image shows a little Mandelbrot set on a vein to a Misiurewicz point, accompanied by little Multibrot sets. The right image illustrates local similarity to a subset of $\mathcal{K}_c$.

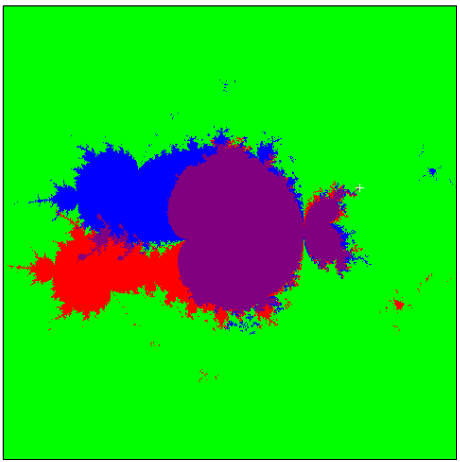
Example 2.3: $f_c(z) = z^3 + cz^2 + \lambda z$ has a persistent Siegel disk for a suitable neutral multiplier λ. Since at least one critical orbit accumulates at the Siegel disk, they cannot both escape. Using a different parametrization with marked critical points, we would have $\mathcal{M}_- \cup \mathcal{M}_+ = \mathbb{C}_*$.

The connectedness locus shows little Mandelbrot sets and copies of quadratic Siegel Julia sets, where the second critical orbit is captured by the Siegel disk. Choosing a little Mandelbrot set on the vein to a Misiurewicz point, an example of local similarity is obtained.



Example 2.4a: An arbitrary one-parameter slice of the two-parameter family of cubic polynomials will not satisfy a persistent critical relation. For simplicity of programming, most examples will be of the form $f_c(z) = c(z^3 - 3z) + b$ with constant b. Here $b = 0.1\,\mathrm{i}$. The images show $\mathcal{M}_+$ in red, $\mathcal{M}_-$ in blue, and their intersection $\mathcal{C}$ is magenta.

The subset in the right image shows the typical splitting of hyperbolic components. Here the split of the blue component happens inside the red component: $+1$ is attracted persistently, and the orbit of -1 may be captured by the attracting basins or not. The split represents their shape qualitatively.



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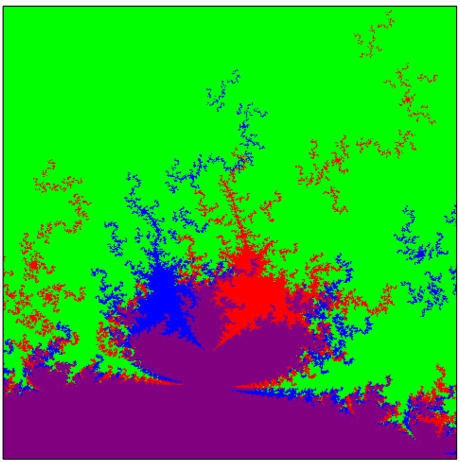
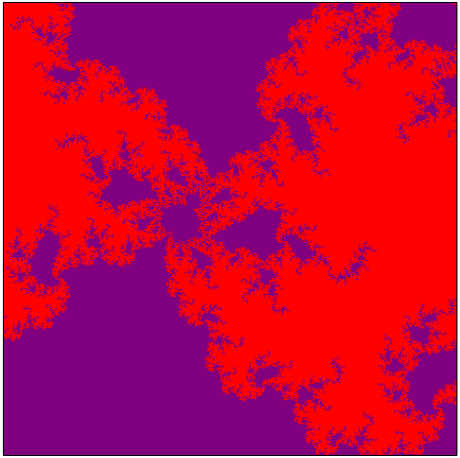
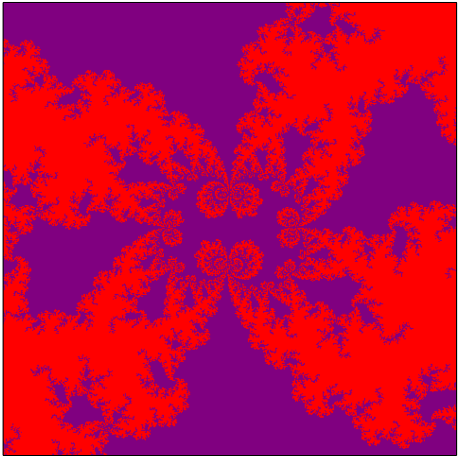


Example 2.4b: The left image shows a little Mandelbrot set in $\mathcal{M}_-$, at the boundary of the split of $\mathcal{M}_-$ in the interior of $\mathcal{M}_+$. At the root of the little Mandelbrot set, the attracting basins show a parabolic implosion, and the filled Julia set $\mathcal{K}_c$ is not moving continuously. Therefore the explanation of splitting is a qualitative one only.





3. Non-degenerate Misiurewicz bifurcation: 1 2 4 5

Dfn.: Consider a one-parameter family of polynomials $f_c(z)$ with a marked critical point ω_c and non-escape locus $\mathcal{M}$. There is a non-degenerate Misiurewicz bifurcation at $c = a$, if

- $f_a^k(\omega_a) = z_a$ is repelling periodic.
- $f_a^k(z)$ is 2:1 at $z = \omega_a$.
- $f_c^k(\omega_c) - z_c$ has a simple zero at $c = a$.

Thm.: Then there are the following similarity phenomena:

- There is a sequence of centers c_n and little Mandelbrot sets $\mathcal{M}_n$, with geometric scaling properties at a and $\partial\mathcal{M}_n \subset \partial\mathcal{M} \subset \mathcal{B}$. [McM]

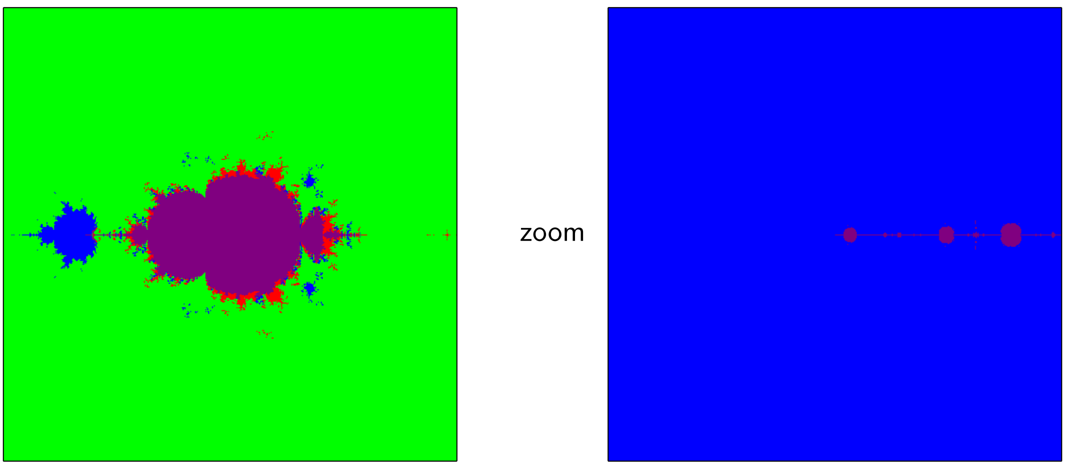
If in addition, $f_a(z)$ has no parabolic cycle, then:

- The non-escape locus $\mathcal{M}$ at $c = a$ is asymptotically similar to the filled Julia set $\mathcal{K}_a$ at any $f_a^m(\omega_a)$, $m \in \mathbb{N}$. [TL]
- There are asymptotic models for $\mathcal{M} - c_n$ on multiple scales.
- There is a local similarity around little Mandelbrot sets.

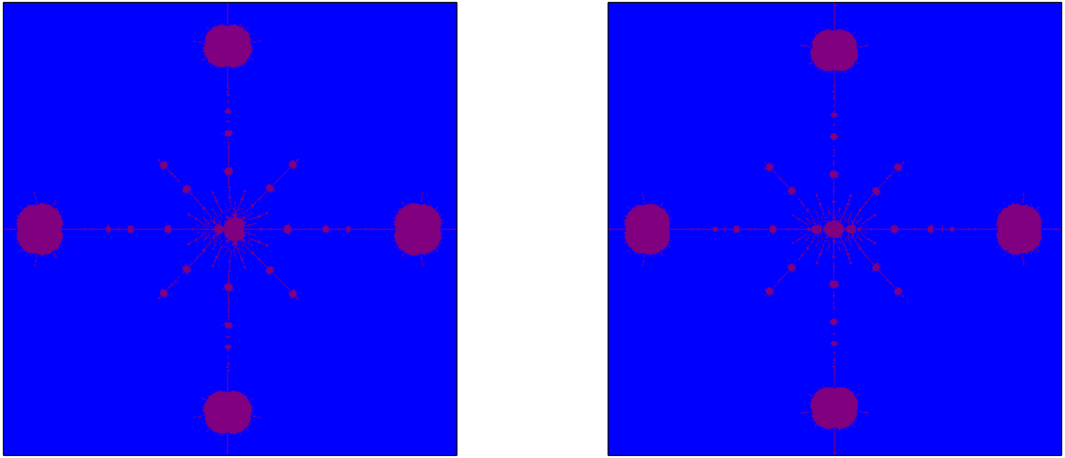
The following examples are of the form $f_c(z) = c(z^3 - 3z) + b$: the filled Julia set is a branch (3.1), fat (3.2), disconnected (3.3), or a dendrite (3.4, 3.5).

Example 3.1a: $f_c(z) = c(z^3 - 3z) + b$ with constant $b = 137/224$. At the Misiurewicz point $c = a = -49/64$, the critcal point ω_+ has preperiod 1 and period 2, while ω_- is attracted. The Misiurewicz point a is located at the tip of the red non-escape locus, inside of the blue cardioid.

In the filled Julia set $\mathcal{K}_a$, preimages of the immediate basin are accumulating at the repelling 2-periodic point. In the red non-escape locus $\mathcal{M}_+$, corresponding capture components are accumulating at the Misiurewicz point $c = a$.

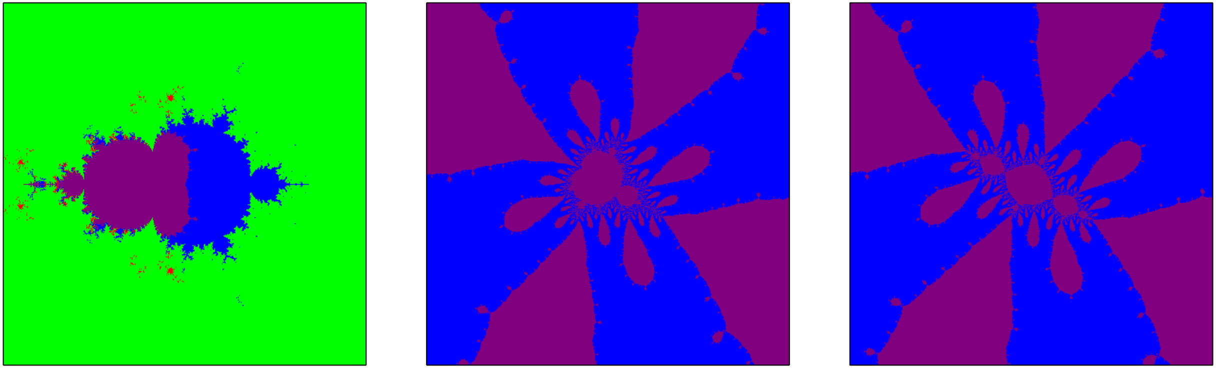


Example 3.1b: Again, $f_c(z) = c(z^3 - 3z) + b$ with constant $b = 137/224$. In the red non-escape locus $\mathcal{M}_+$, little Mandelbrot sets approaching the Misiurewicz point $c = a = -49/64$ show capture components in their decorations. There is a local similarity to the decorations of the little Julia set, which contain preimages of the immediate basin.



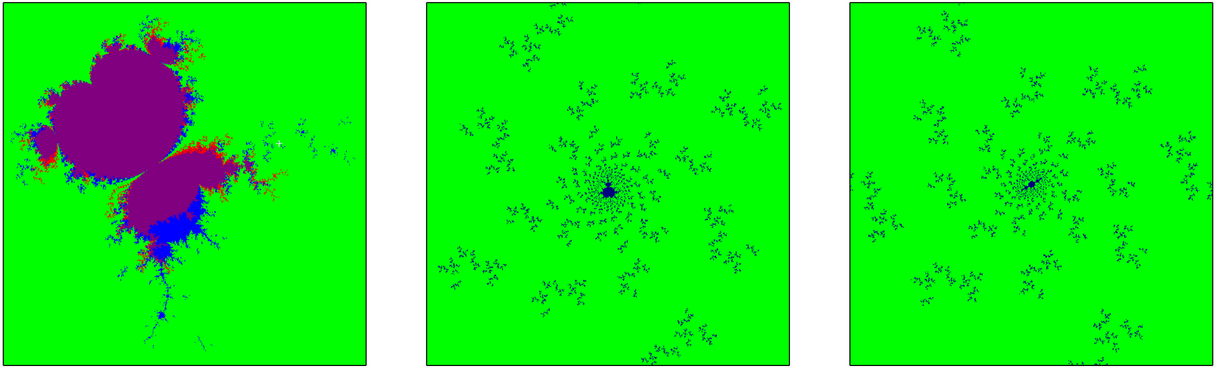
Example 3.2: $f_c(z) = c(z^3 - 3z) + b$ with constant $b = -1.5$. At the Misiurewicz point $c = a = 0.25$, the critcal point ω_+ has preperiod 1 and period 1, while ω_- is attracted to a fixed point. The repelling fixed point $z_a = f_a(\omega_+)$ is located on the boundary of the attracting basin, and the Misiurewicz point a is located on the boundary of the red non-escape locus $\mathcal{M}_+$. (Cf. the parabolic Example 4.3.)

$\mathcal{M}_+$ shows the linear asymptotic self-similarity at $c = a$. It contains little Mandelbrot sets approaching a from above, decorated with capture components. The right image illustrates local similarity to the filled Julia set $\mathcal{K}_c$.



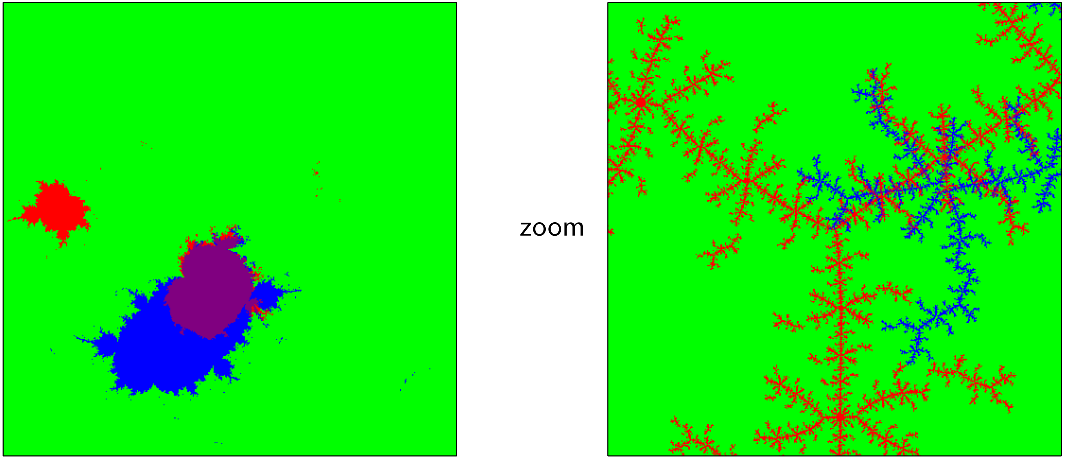
Example 3.3: $f_c(z) = c(z^3 - 3z) + b$ with constant $b \approx -1.58365 + \mathrm{i}\,2.99497$. At the Misiurewicz point $c = a \approx 0.11058 - \mathrm{i}\,0.02769$, the critcal point ω_- has preperiod 2 and period 2, while ω_+ is escaping to ∞. Locally, the red non-escape locus $\mathcal{M}_+$ is empty, and the blue non-escape locus $\mathcal{M}_-$ is asymptotically similar to the disconnected filled Julia set $\mathcal{K}_a$ at a repelling 2-periodic point z_a. (Note that $\mathcal{B} = \partial\mathcal{M}_-$ locally.)

The middle and right images illustrate local similarity between the disconnected decorations of a little Mandelbrot set in $\mathcal{M}_-$ and a corresponding little Julia set in $\mathcal{K}_c$.



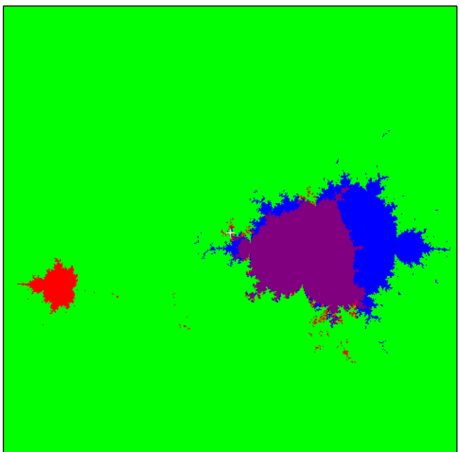
Example 3.4: $f_c(z) = c(z^3 - 3z) + b$ with constant $b \approx -0.47278 + \mathrm{i}\,0.87619$. At the Misiurewicz point $c = a \approx 0.16782 + \mathrm{i}\,0.29890$, both critical points of $f_a(z)$ are preperiodic. ω_- has preperiod 1 and period 2, and ω_+ has preperiod 4 and period 1.

The filled Julia set $\mathcal{K}_a$ is a dendrite. Both the blue non-escape locus $\mathcal{M}_-$ and the red non-escape locus $\mathcal{M}_+$ are asymptotically similar to $\mathcal{K}_a$ at different periodic points. The connectedness locus $\mathcal{C} = \mathcal{M}_- \cap \mathcal{M}_+$ is more involved.



Example 3.5: $f_c(z) = c(z^3 - 3z) + b$ with constant $b \approx -1.33742 - \mathrm{i}\,0.27217$. At the Misiurewicz point $c = a \approx -0.33129 + \mathrm{i}\,0.13609$, both critical points of $f_a(z)$ are preperiodic. They are mapped to the same repelling 2-cyle: ω_- in 5 iterations, and ω_+ in 4 iterations. The filled Julia set $\mathcal{K}_a$ is a dendrite.

Both the blue non-escape locus $\mathcal{M}_-$ and the red non-escape locus $\mathcal{M}_+$ are asymptotically self-similar with the same scaling factor ρ_a. They are similar to the filled Julia set $\mathcal{K}_a$ at the same 2-periodic point z_a, but with different similarity factors. The connectedness locus $\mathcal{C} = \mathcal{M}_- \cap \mathcal{M}_+$ is more involved.



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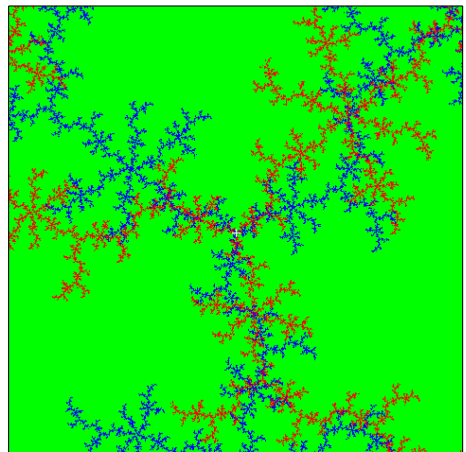


4. Degenerate and parabolic examples:

What about asymptotic similarity, when some condition from Section 3 is not satisfied?

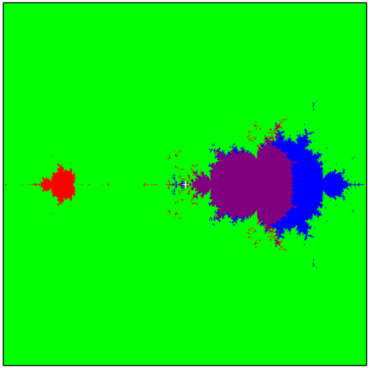
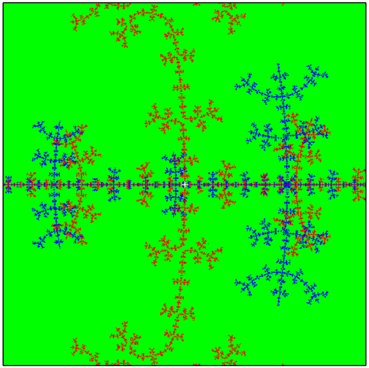
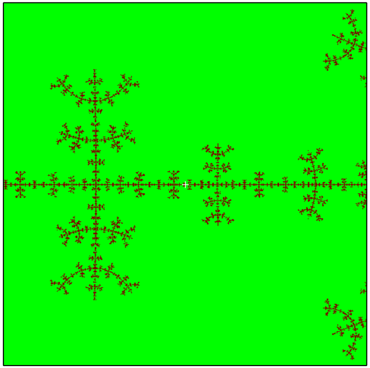
Example 4.1: $f_c^k(\omega_c) - z_c$ has a double zero at $c = a$, thus $u'(a) = 0$.

Example 4.2: At $c = a$, $f_a(z)$ maps $\omega_- \mapsto \omega_+$, thus $f_a^k(z)$ is 4:1 at $z = \omega_-$.

Examples 4.3 and 4.4: $f_a(z)$ has a parabolic cycle.

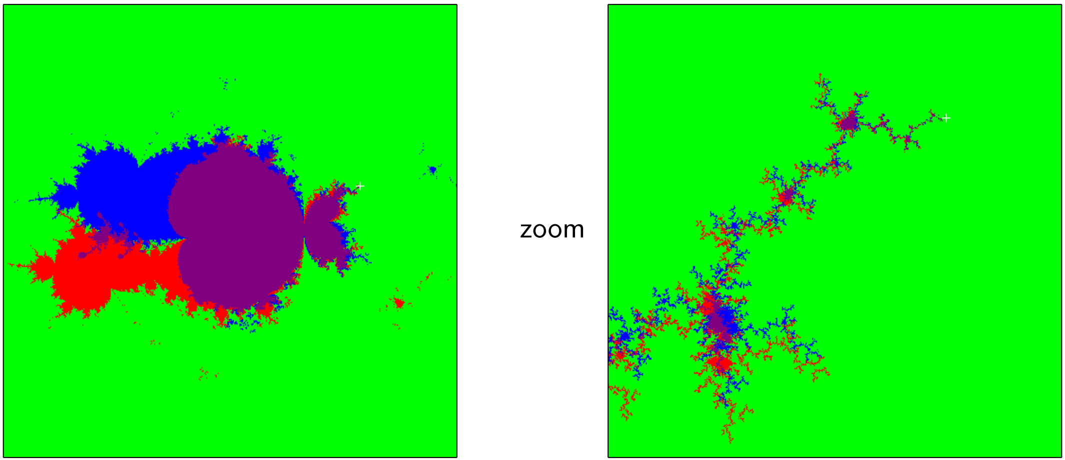
Example 4.1: $f_c(z) = c(z^3 - 3z) + b$ with constant $b = 1 - 1.5\sqrt[3]{4}$. At the Misiurewicz point $c = a = -0.25\sqrt[3]{4}$, the critical point ω_+ has preperiod 1 and period 2. But $f_c^k(\omega_c) - z_c$ has a double zero at $c = a$, thus $u'(a) = 0$.

The filled Julia set $\mathcal{K}_a$ in the right image has two branches at the repelling 2-periodic point $z = z_a$, but the red non-escape locus $\mathcal{M}_+$ has 4 branches at the Misiurewicz point $c = a$. It is asymptotically similar to $\sqrt{X_a}$ and to $\sqrt{\mathcal{K}_a - z_a}$.

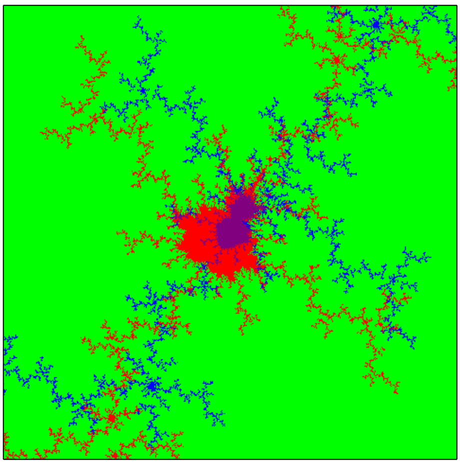
Example 4.2a: $f_c(z) = c(z^3 - 3z) + b$ with constant $b \approx -0.11613 - \mathrm{i}\,0.37269$. At the Misiurewicz point $c = a \approx 0.55806 + \mathrm{i}\,0.18634$, the critical point ω_- is mapped to the other critical point ω_+, which is mapped to a repelling fixed point in 2 iterations. Thus ω_- has preperiod 3 and period 1, but $f_a^3(z)$ is 4:1 at $z = \omega_-$. (This behavior was persistent in Example 2.2, but here the critical orbits separate for $c \neq a$.)

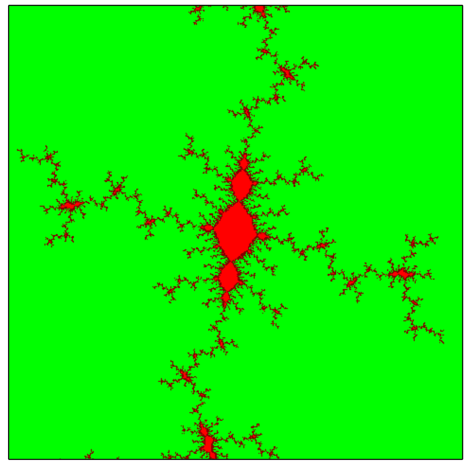
Close to the Misiurewicz point $c = a$, both connectedness loci show incomplete little Multibrot sets. There cannot be a complete little Multibrot set, since there is no persistently degenerate critical point of some iterate. But little Mandelbrot sets will be dense in the bifurcation locus.



Example 4.2b: Consider again $f_c(z) = c(z^3 - 3z) + b$ with constant $b \approx -0.11613 - \mathrm{i}\,0.37269$ and a subset around a little Mandelbrot set in the red non-escape locus of ω_+, close to the Misiurewicz point $c = a \approx 0.55806 + \mathrm{i}\,0.18634$. Its decorations show local similarity to the decorations in a corresponding little Julia set.

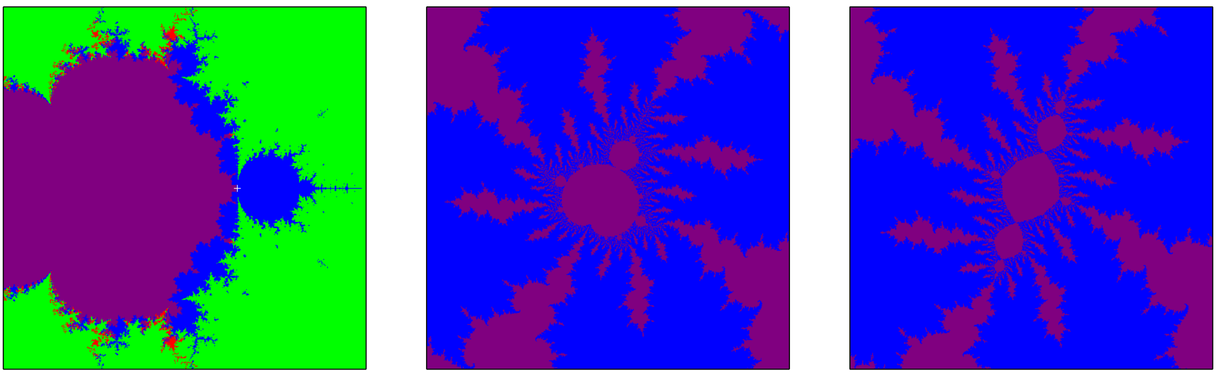
The blue non-escape locus of ω_- must contain little Mandelbrot sets as well. Its bigger components look like incomplete Multibrot sets or embedded Julia sets. Here the big component means that $f_c(\omega_-)$ belongs to the little Julia set, which is not moving continuously.



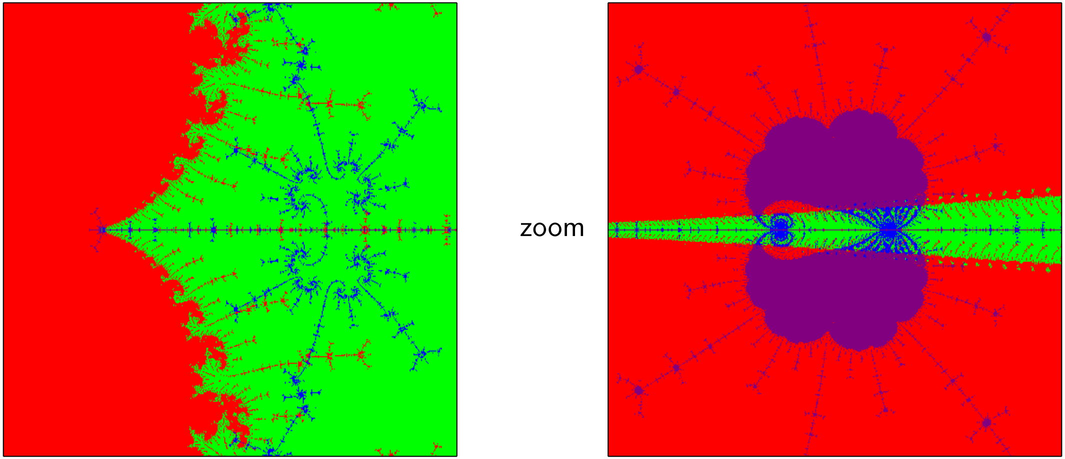


Example 4.3: $f_c(z) = c(z^3 - 3z) + b$ with constant $b = -10/9$. At the Misiurewicz point $c = a = 4/9$, the critical point ω_+ has preperiod 1 and period 1. The repelling fixed point z_a is on the boundary of a basin as in Example 3.2, but now it is the basin of a parabolic fixed point. So the filled Julia set $\mathcal{K}_c$ does not depend continuously on the parameter $c \approx a$.

But the red non-escape locus of ω_+ seems to approach $c = a$ from a sector inside the blue hyperbolic component, where $\mathcal{K}_c$ is continuous in fact. So $\mathcal{M}_+$ will be asymptotically similar to $\mathcal{K}_a$ and there seems to be local similarity as well.



Example 4.4: $f_c(z) = c(z^3 - 3z) + b$ with constant $b \approx -2.06113$. At the Misiurewicz point $c = a \approx -0.33426$, the critical point ω_- has preperiod 1 and period 2. Since $f_a(z)$ has a parabolic 3-cycle as well, the filled Julia set $\mathcal{K}_a$ does not move continuously for $c \approx a$. In $\mathcal{K}_a$, the repelling 2-periodic point z_a is approached by filled cauliflowers. In the blue non-escape locus of ω_-, the Misiurewicz point $c = a$ is approached by a kind of imploded cauliflowers. However, these look imploded mostly in a small slit, so they might be converging to a filled cauliflower in fact, and asymptotic similarity might be true in spite of the parabolic cycle.



5. Rational families:

Suppose that $f(z)$ is a rational map with an attracting periodic point α. Define the filled Julia set $\mathcal{K} = \{z \,|\, f^n(z) \not\to \alpha\}$. Its boundary $\partial\mathcal{K}$ is the Julia set.

For a one-parameter family $f_c(z)$ of rational maps, such that α_c is either persistently attracting, or attracting for parameters $c \approx a$, consider the filled Julia sets $\mathcal{K}_c = \{z \,|\, f_c^n(z) \not\to \alpha_c\}$. If the critical points are marked, define the non-escape loci $\mathcal{M}^j = \{c \,|\, \omega_c^j \in \mathcal{K}_c\} = \{c \,|\, f_c^n(\omega_c^j) \not\to \alpha_c\}$. The bifurcation locus is given by $\mathcal{B} = \cup\, \partial\mathcal{M}^j$.

If for $c = a$, a non-degenerate Misiurewicz bifurcation happens for a critical point ω_c^j, and if $f_a(z)$ has no neutral cycle, then $\mathcal{M}^j - a$ and $\mathcal{K}_a - f_a^k(\omega_a^j)$ will be asymptotically similar. Sequences of little Mandelbrot sets show asymptotic similarity on multiple scales and local similarity.

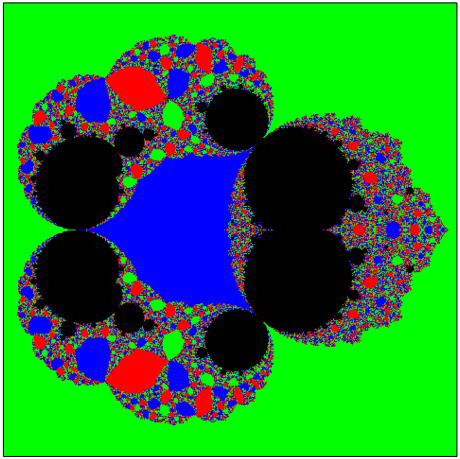
Example 5.1: quadratic rational maps with superattracting 3-cycle.

Example 5.2: arbitrary slice of quadratic rational maps.

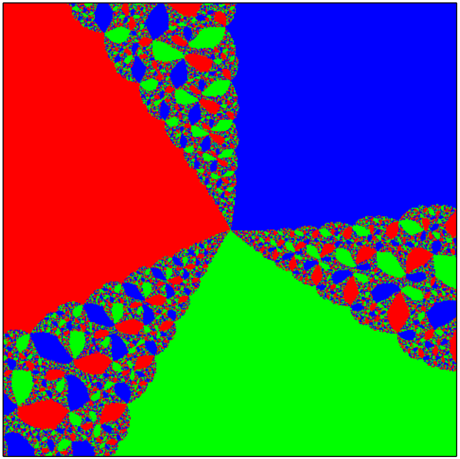
Example 5.3: cubic Newton method.

Example 5.4: quartic Newton method with persistently superattracting 2-cycle.

Example 5.1a: This family of rational maps $f_c(z) = \frac{z^2+A_c}{z^2+B_c}$ has a persistently superattracting 3-cycle containing $\omega_c^2 = \infty$. At the Misiurewicz point $c = a \approx -0.07629 + \mathrm{i}\, 1.32623$, the free critical point $\omega_c^1 = 0$ has preperiod 4 and period 1. The right image shows the asymptotic self-similarity of the non-escape locus $\mathcal{M}^1$ at $c = a$.

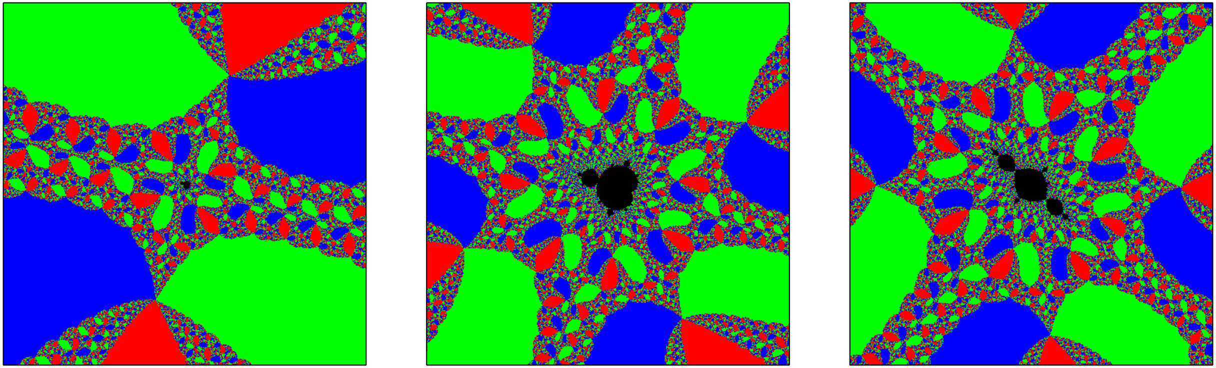


zoom



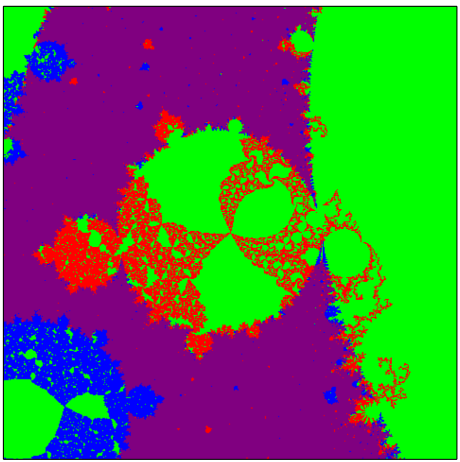
Example 5.1b: Consider again the family of rational maps $f_c(z) = \frac{z^2+A_c}{z^2+B_c}$ having a persistently superattracting 3-cycle containing $\omega_c^2 = \infty$. There is a little Mandelbrot set of period 24 close to the Misiurewicz point $c = a \approx -0.07629 + \mathrm{i}\,1.32623$ of preperiod 4 and period 1.

The images illustrate the asymptotic models for the non-escape locus $\mathcal{M}^1$ of the free critical point $\omega_c^1 = 0$, and its local similarity to the filled Julia set $\mathcal{K}_c$ around the little Julia set.

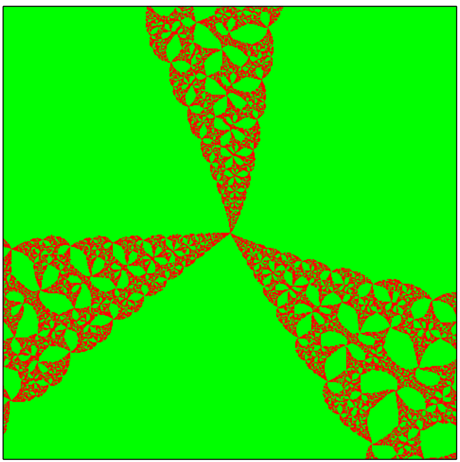


Example 5.2a: Consider the one-parameter family of quadratic rational maps $f_c(z) = \frac{z^2+c}{z^2+b}$ with constant $b \approx 1.75307 + \mathrm{i}\, 0.20227$. This is an arbitrary slice through the 2-dimensional parameter space, i.e., the critical orbits are independent. At the Misiurewicz point $c = a \approx -0.52181 - \mathrm{i}\, 3.63579$, the map is the same as in the previous example: the critical point $\omega_a^1 = 0$ has preperiod 4 and period 1, and the critical point $\omega_a^2 = \infty$ is 3-periodic. For parameters $c \approx a$, there will be an attracting 3-periodic point $\alpha_c \approx \infty$, which is used to define the filled Julia set $\mathcal{K}_c$ and the non-escape locus $\mathcal{M}^1$.

The non-escape locus $\mathcal{M}^1$ of $\omega_c^1 = 0$ shows asymptotic self-similarity and similarity to the filled Julia set. (The bifurcation locus $\partial\mathcal{M}^1$ is red and $\partial\mathcal{M}^2$ is blue.)

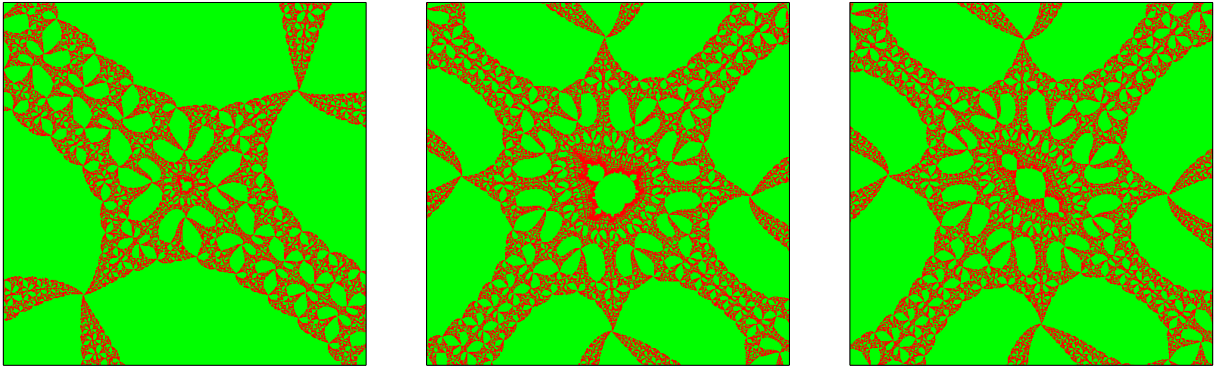


zoom



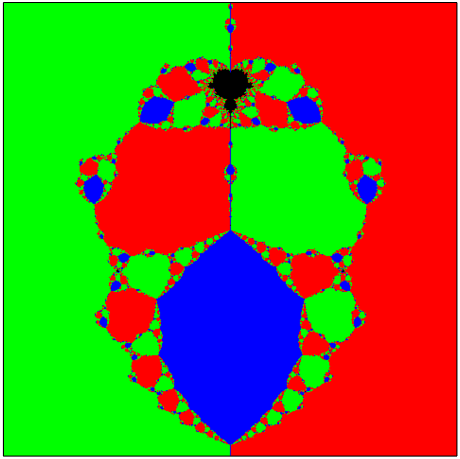
Example 5.2b: Again, consider the one-parameter family of quadratic rational maps $f_c(z) = \frac{z^2+c}{z^2+b}$ with constant $b \approx 1.75307 + \mathrm{i}\, 0.20227$ and a little Mandelbrot set in the non-escape locus $\mathcal{M}^1$ of $\omega_c^1 = 0$ close to the Misiurewicz point $c = a \approx -0.52181 - \mathrm{i}\, 3.63579$.

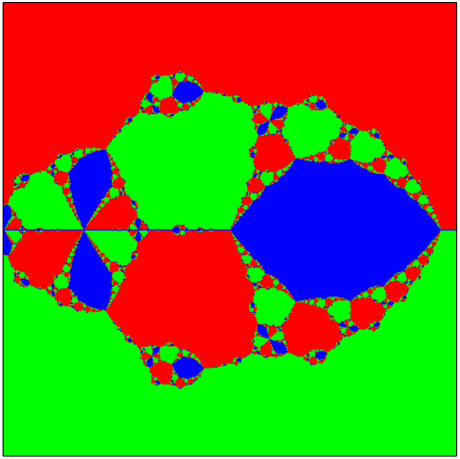
The images illustrate the asymptotic models for the non-escape locus $\mathcal{M}^1$ and its local similarity to the filled Julia set $\mathcal{K}_c$ around the little Julia set.
(The bifurcation locus $\partial\mathcal{M}^1$ is red and some of the green components belong to $\mathcal{M}^1$, some to the escape locus. $\omega_a^2 = \infty$ is attracted to the 3-cycle, thus the blue $\partial\mathcal{M}^2$ does not appear here.)



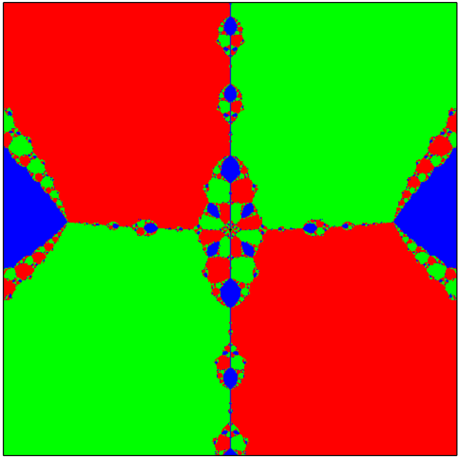
Example 5.3a: Consider the family $f_c(z) = z - P_c(z)/P_c'(z)$ of cubic Newton methods, where $P_c(z)$ has the roots 1 and $-0.5 \pm c$. The right image shows the attracting basins in different colors, and the filled Julia set $\mathcal{K}_c$ in black. The left image shows the parameter plane: the colors indicate that the free critical point $\omega = 0$ is attracted to a root, and its non-escape locus $\mathcal{M}$ is black.

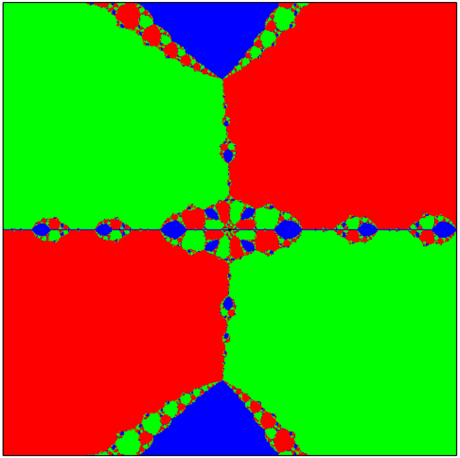
At the Misiurewicz point $c = a$, $\omega = 0$ goes to the repelling fixed point ∞ in 4 iterations. Observe the asymptotic similarity between the non-escape locus $\mathcal{M}$ and the filled Julia set $\mathcal{K}_a$.



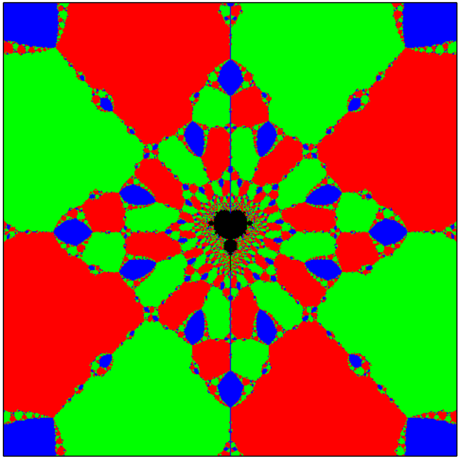


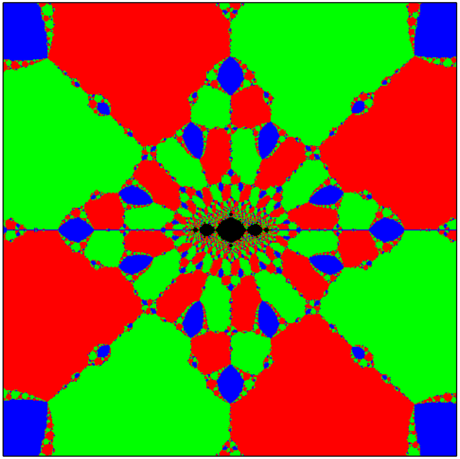
Example 5.3b: At the same Misiurewicz point $c = a$ in the family of cubic Newton maps, observe the asymptotic models of the first level: the left image shows the non-escape locus $\mathcal{M}$ around a little Mandelbrot set, and the right image shows the corresponding part of the filled Julia set $\mathcal{K}_c$ around the little Julia set.





Example 5.3c: At the same Misiurewicz point $c = a$ in the family of cubic Newton maps, observe the local similarity between the non-escape locus $\mathcal{M}$ of $\omega = 0$ around a little Mandelbrot set, and the corresponding part of the filled Julia set $\mathcal{K}_c$ around the little Julia set.





Example 5.4: Quartic Newton maps form a two-parameter family. Consider the one-parameter family $f_c(z)$ having a persistently superattracting 2-cycle. There is one free critcal point ω_c remaining. Define the filled Julia set $\mathcal{K}_c$ and the non-escape locus $\mathcal{M}$ of ω_c by non-convergence to any of the four superattracting fixed points. Capture zones of the 2-cycle are contained in $\mathcal{M}$. At the Misiurewicz point $z = a \approx -2.06853 + \mathrm{i}\,2.20009$, ω_a goes to the repelling fixed point ∞ in 4 iterations.

Observe the asymptotic self-similarity of $\mathcal{M}$ at a, and the local similarity of $\mathcal{M}$ around a little Mandelbrot set of period 8 and $\mathcal{K}_c$ around a little Julia set.