Combinatorics, external rays, and twisted polynomials.

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1. Background.
2. The Stripping Algorithm.
3. Early returns.
4. Non-admissible combinatorics.
5. The proof.
6. Recapture of the critical point.

The images were made with Mandel, a program available from www.mndynamics.com .

Talk given at the conference on dynamical systems in Göttingen, August 2011.

1. Background:

Iteration of $f_c(z) = z^2 + c$.

Filled Julia set $\mathcal{K}_c = \{z \in \mathbb{C} \mid f_c^n(z) \not\to \infty\}$.

Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} \mid c \in \mathcal{K}_c\}$.

Main tool: External rays with rational angles. Defined by conformal mappings $\Phi_c : \widehat{\mathbb{C}} \setminus \mathcal{K}_c \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\Phi_M : \widehat{\mathbb{C}} \setminus \mathcal{M} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Applications include: description of points and subsets; topological models.

Other combinatorial concepts: Hubbard trees, kneading sequences, and internal addresses.

1.1 External rays

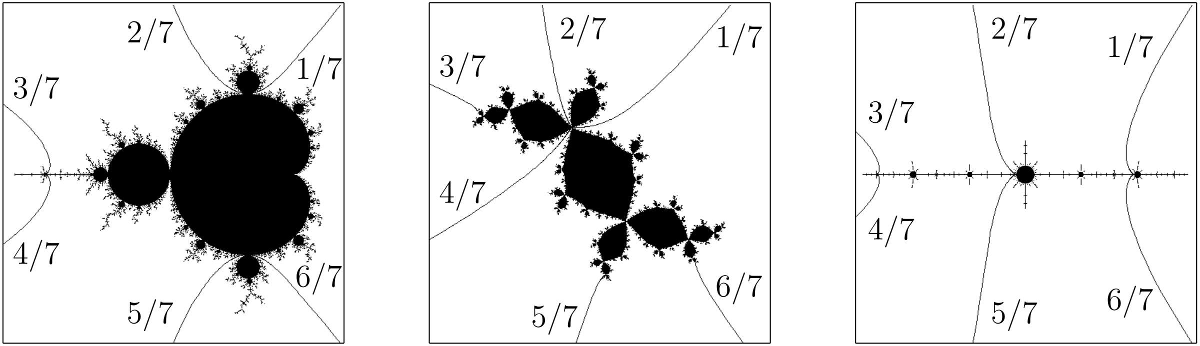
Polynomial $f_c(z)$ corresponds to angle doubling modulo 1.
Symbolic dynamics of binary digits.

Landing of dynamic rays with rational angle θ:

Odd denominator — periodic digits — landing at periodic point.

Even denominator — preperiodic digits — landing at preperiodic point.

Landing pattern is stable except for bifurcations; Mandel demo 3, p. 5, 6, 10.
Parameter rays are landing at roots and Misiurewicz points.



1.2 Hubbard tree

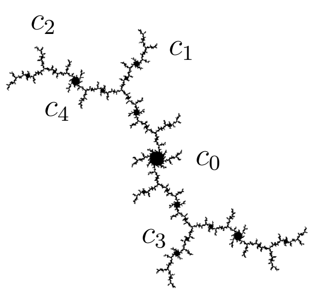
Postcritically finite polynomial f_c; here 5-periodic.

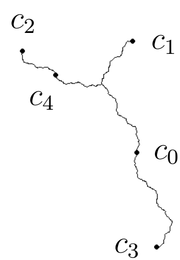
Concrete Hubbard tree: connect postcritical points (and 0) by arcs within $\mathcal{K}_c$.

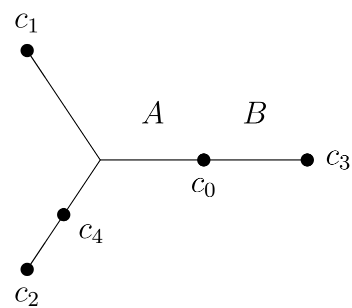
Oriented Hubbard tree: considered as a planar graph.

There are two components A and B at the critical point $z = 0$.

Equivalently: the critical value $z = c$ is an endpoint.







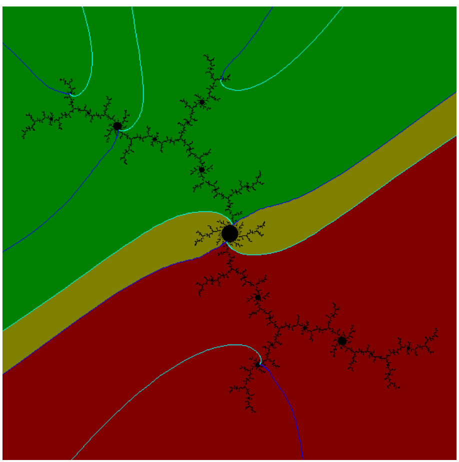
1.3 Kneading sequence

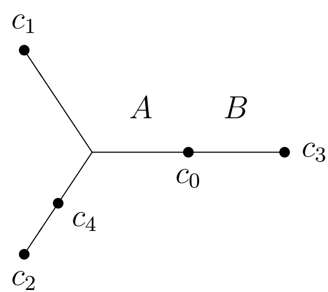
Symbolic dynamics of sequence $\overline{AABA*}$.

j-th symbol says whether $f_c^{j-1}(c)$ is in A, in B, or $=0$.

Or for angle θ: A corresponds to $2^{j-1}\theta \in (\frac{\theta}{2}, \frac{\theta+1}{2})$ ...

Both characteristic angles, here $\theta_- = \frac{5}{31}$ and $\theta_+ = \frac{6}{31}$, give the same kneading sequence: the yellow strip is forbidden.





1.4 Internal address

The internal address is describing a kneading sequence by increasing periods. These correspond to hyperbolic components in $\mathcal{M}$, where the kneading sequence is changing.

Example: $\overline{AABA*}$ is obtained by changing $\overline{A} \to \overline{AAB} \to \overline{AABA*}$, so the internal address is 1-3-5.

Conversely, the internal address 1-3-5 gives $\overline{A} \to \overline{AAB} \to \overline{AABA*}$.

Further applications of internal addresses and kneading sequences include:

A combinatorial proof that periodic parameter rays are landing in pairs at roots (Schleicher).

Homeomorphisms between sublimbs (Dudko–Schleicher).

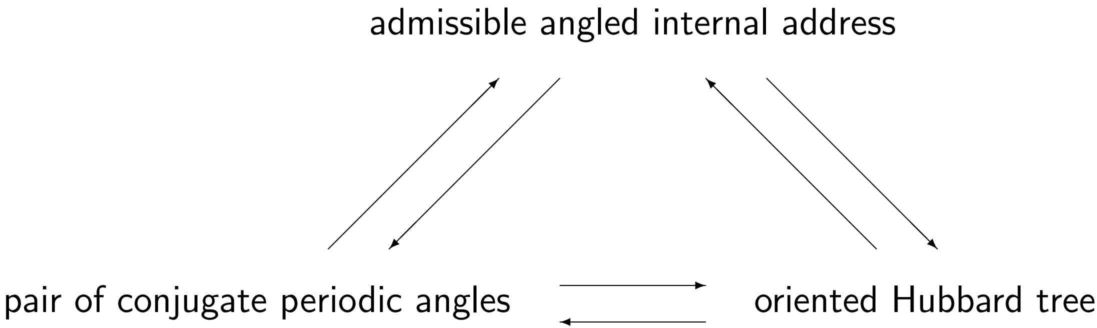
1.5 Concrete and abstract combinatorial concepts

$$\left.\begin{array}{r}\text{hyp. component, center, root}\\[2ex] \text{pair of conjugate per. angles}\\[2ex] \text{oriented Hubbard tree}\end{array}\right\} \quad \to \quad \left\{\begin{array}{l}\text{finite internal address}\\[2ex] \ast\text{-periodic kneading sequence}\\[2ex] \text{abstract Hubbard tree}\end{array}\right.$$

The mapping is not injective: the same addresses are realized in sublimbs of equal denominators.

The mapping is not surjective: an internal address may be non-admissible.

1.6 Conversion algorithms:



Several algorithms for conversions between different combinatorial descriptions are given by Bruin and Schleicher.

The algorithms converting an internal address or kneading sequence to external angles are quite involved: growing of trees or constructing the Hubbard tree first.

2. The Stripping Algorithm:

For a given kneading sequence and (angled) internal address, the characteristic angles shall be computed.

This is done recursively for all periods in the internal address.

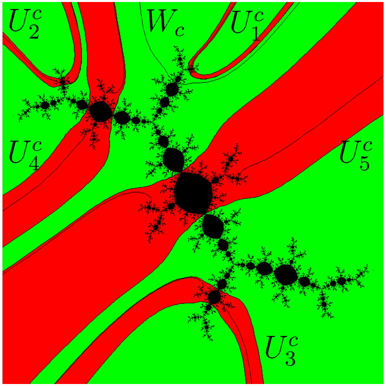
In the step from 1-. . . -k to 1-. . . -k-n, suppose that the center $\hat{c}$ of period n belongs to the p/q-sublimb of the center c_0 of period k. Then the angles of period qk are determined from those of period k by Douady tuning.

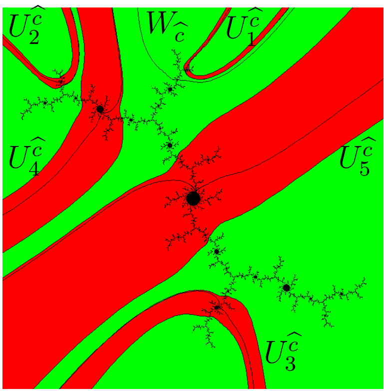
If $n < qk$, $\hat{c}$ will be behind the satellite component. Its angles are computed by iterating backwards.

2.1 Example of 1-3-5 or $\overline{AABA*}$ with strips:

1-3 has the angles $\frac{1}{7}$ and $\frac{2}{7}$. Now $q = 2$, and 1-3-6 has angles $\frac{10}{63}$ and $\frac{17}{63}$.

Compare the dynamics of c with period 6 to that of $\widehat{c}$ with period 5:
5-periodic rays of $\widehat{c}$ are not yet landing together for c.
6-periodic rays of c are still landing together for $\widehat{c}$.
Preimages of 6-periodic rays define stable strips in both planes. They are chosen according to the kneading sequence of $\widehat{c}$. U_1 gives $\frac{5}{31}$ and $\frac{6}{31}$.





2.2 The Stripping Algorithm with strips:

For the step from 1-...-k to 1-...-k-n, start with the center c and the angles $\theta_\pm$ of period qk .

0. The wake W is bounded by the external rays for $\theta_\pm$.
1. Consider the strip $U_n = f_c^{-1}(W)$. Set $\mathbb{C} \setminus \overline{U_n} = A \cup B$.
2. For $j = n-1$ down to $j = 1$, first check if $U_{j+1} \subset W$. Then the algorithm will be branching. W is replaced with subsets of $W \setminus \overline{U_{j+1}}$.
3. For $j = n-1$ down to $j = 1$, then choose a component of $f_c^{-1}(U_{j+1})$ for U_j according to the kneading sequence of 1-...-k-n.
4. A branch is failing, if $U_1 \not\subset W$. Otherwise the unique n-periodic rays with angles $\Theta_\pm$ are recorded.

2.3 The Stripping Algorithm with angles:

For the step from 1-. . .-k to 1-. . .-k-n, start with the center c and the angles $\theta_\pm$ of period qk .

0. The wake W corresponds to the interval $(\theta_-\,,\,\theta_+)$.
1. In step n, compute the four angles $\Theta_1 = \theta_-/2$, $\Theta_2 = \theta_+/2$, $\Theta_3 = (\theta_- + 1)/2$, and $\Theta_4 = (\theta_+ + 1)/2$.
2. In steps $j = n - 1$ down to $j = 1$, first check if $\theta_- < \Theta_1 < \theta_+$. Then the algorithm is branching: (θ_-, θ_+) is replaced with $(\theta_-, \min\{\Theta_1, \Theta_3\})$ in one branch and with $(\max\{\Theta_2, \Theta_4\}, \theta_+)$ in the other branch.
3. In steps $j = n-1$ down to $j = 1$, then compute preimages. Case A: Θ_i is replaced with $(\Theta_i+1)/2$ if $0 < \Theta_i < \theta_-$ and with $\Theta_i/2$ if $\theta_+ < \Theta_i < 1$. It is vice versa in case B.
4. The branch is not failing if $\theta_- < \Theta_1 < \theta_+$. Then there are two unique n-periodic angles $\Theta_\pm$ with $\Theta_1 < \Theta_\mp < \Theta_2$ and $\Theta_3 < \Theta_\pm < \Theta_4$.

2.4 Example of 1-3-5 or $\overline{AABA*}$ with angles:

$(\frac{10}{63}, \frac{17}{63})$	W	$[\frac{5}{31}, \frac{8}{31}]$
$(\frac{5}{63}, \frac{17}{126}) \cup (\frac{73}{126}, \frac{40}{63})$	U_5	$[\frac{3}{31}, \frac{4}{31}] \cup [\frac{18}{31}, \frac{19}{31}]$
$(\frac{34}{63}, \frac{143}{252}) \cup (\frac{73}{252}, \frac{20}{63})$	U_4	$[\frac{17}{31}, \frac{17}{31}] \cup [\frac{9}{31}, \frac{9}{31}]$
$(\frac{97}{126}, \frac{395}{504}) \cup (\frac{325}{504}, \frac{83}{126})$	U_3	$[\frac{24}{31}, \frac{24}{31}] \cup [\frac{20}{31}, \frac{20}{31}]$
$(\frac{97}{252}, \frac{395}{1008}) \cup (\frac{325}{1008}, \frac{83}{252})$	U_2	$[\frac{12}{31}, \frac{12}{31}] \cup [\frac{10}{31}, \frac{10}{31}]$
$(\frac{97}{504}, \frac{395}{2016}) \cup (\frac{325}{2016}, \frac{83}{504})$	U_1	$[\frac{6}{31}, \frac{6}{31}] \cup [\frac{5}{31}, \frac{5}{31}]$

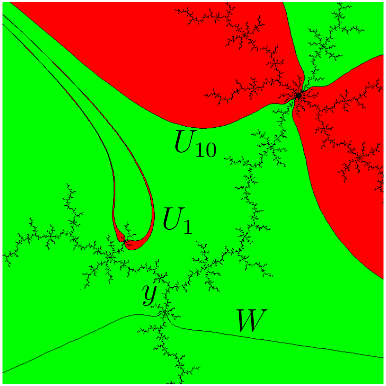
The intervals of U_1 are approximately $(\frac{5.97}{31}, \frac{6.07}{31}) \cup (\frac{4.998}{31}, \frac{5.11}{31})$, determining unique angles $\Theta_+ = \frac{6}{31}$ and $\Theta_- = \frac{5}{31}$ of period 5.

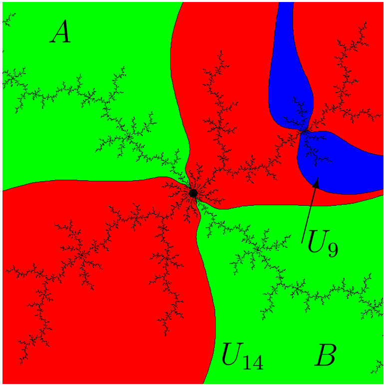
In the third column, only 5-periodic angles are used, rounded to the interior of the intervals when necessary.

The first example is given by 1-2-6-7-13-14 or $\overline{ABABAABABABAB*}$. Here $U_{10} \subset W$, so its preimages belong to U_{14} instead of A and B.

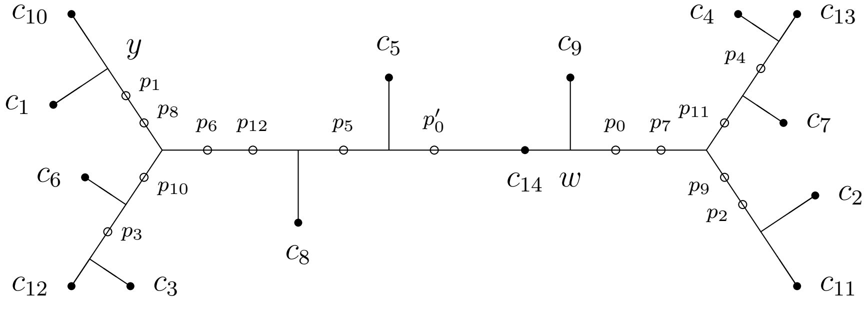
The Stripping Algorithm is branching to try both strips possible for U_9. The first branch is failing with the angles $\frac{6443}{16383}$ and $\frac{6444}{16383}$; the second branch is succeeding with the angles $\frac{6347}{16383}$ and $\frac{6348}{16383}$.

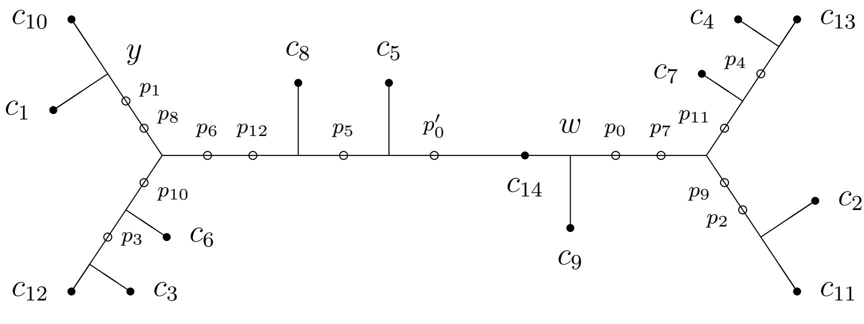
This phenomenon requires much consideration in the algorithm and the proof.



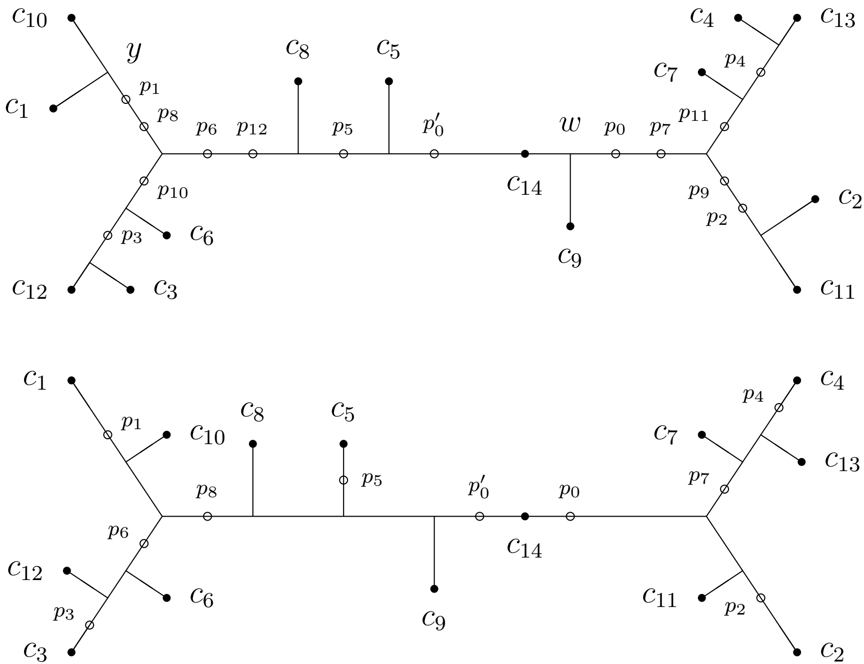


3.1 Oriented Hubbard trees; the correct branch with the angles $\frac{6347}{16383}$ and $\frac{6348}{16383}$, and the failing branch:





3.2 Oriented Hubbard trees; the failing branch and a similar Hubbard tree for a different center. It has the external angles $\frac{6443}{16383}$ and $\frac{6444}{16383}$. In the kneading sequence, there is an A instead of a B at the position of c_9:

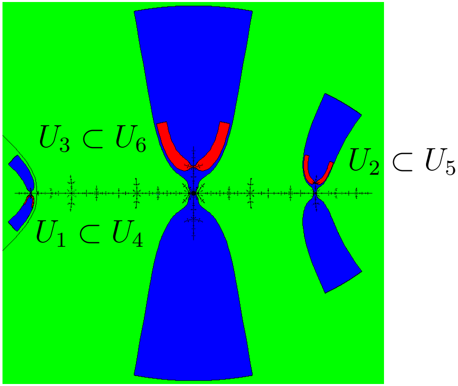


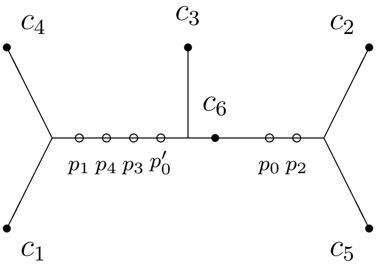
4. The non-admissible example of 1-2-4-5-6 or $\overline{ABAAB*}$

This combinatorics is not admissible: there is no oriented Hubbard tree, external angle, or quadratic polynomial realizing these combinatorics.

Left: the Stripping Algorithm has an early return $U_4 \subset W$. Both branches are failing, since U_1 turns up on the wrong side of $W \setminus \overline{U_4}$.

Right: the abstract Hubbard tree has a 3-periodic evil branch point.





5. The proof of the Stripping Algorithm shows:

1. Iterating the strips U_j backwards is well-defined, since $c \notin U_{j+1}$. No strip is linked with the (possibly reduced) wake W.

2. If the algorithm does not branch, or if a branch does not fail, U_1 determines the desired characteristic angles of period n. Any other branch will be failing. The address 1-$\ldots$-k-n is realized.

3. If the algorithm is branching and all branches are failing, the address 1-$\ldots$-k-n will not be realized by a quadratic polynomial.

5.1 Pulling back is well-defined:

Show that no strip is containing c. Strips are disjoint or nested.

If there are several early returns, the earlier ones will not be before later ones; therefore they will not be linked with a reduced wake.

5.2 Non-failing branch gives correct angles:

In any case, comparing itineraries shows that $U_1 \subset W_0$ (original wake). So there are angles of period dividing n.

Forward iterates of rays are contained in the strips U_j. So the kneading sequence is the required one. In the case of an early return, note the components besides $f_c^{-1}(W) \subset U_n$ defining the kneading sequence. If the branch is not failing, the strips will be on the correct side.

5.3 Admissible address gives non-failing branch:

Show that the strips U_j, defined originally for c, do not bifurcate for parameters between c and $\widehat{c}$.

This is obvious when there is no early return, since a dynamic ray is bifurcating only when the critical value belongs to an image of this ray.

If there are early returns, there will be a unique branch point y between c and z_1, which corresponds to a Misiurewicz point a between c and $\widehat{c}$. All possible image rays belong to other branches of y or a, respectively.

6. Recapture of the critical point:

Recall the example of the center $\widehat{c}$ of period 5 behind the center c of period 6: in the latter plane, the preimages $z_j \in U_j$ represent the critical orbit of $f_{\widehat{c}}$, except that $f_c(0) = c$ so $f_c(0) \neq z_1$.

Connecting $z_1, \ldots, z_5$ by a tree $T \subset \mathcal{K}_c$ gives the oriented Hubbard tree of $f_{\widehat{c}}$, but the mapping $f : T \to T$ is not the restriction of f_c: it must be modified in a neighborhood of 0.

In fact this modification turns f_c into a topological polynomial g, which will be Thurston equivalent to $f_{\widehat{c}}$.

6.1 Recapture using Thurston Theorem

Dfn: Consider $f_c(z) = z^2 + c$, and a preimage $z_1 \neq c$ of 0, such that z_1, $z_2 = f_c(z_1)$, ... , $z_n = f_c^{n-1}(z_1) = 0$ are distinct. Set $\mathcal{P} = \{z_1, z_2, \ldots, z_n, \infty\}$. Suppose γ is an arc from c to z_1, which is disjoint from $\mathcal{P} \setminus \{z_1\}$. **Recapture** along γ gives a center $\hat{c}$ of period n as follows:

1. Choose a disk neighborhood Δ of γ disjoint from $\mathcal{P} \setminus \{z_1\}$.

2. Choose a homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\phi(c) = z_1$, which is the identity on $\widehat{\mathbb{C}} \setminus \Delta$.

3. The topological polynomial $g = \phi \circ f_c$ is combinatorially equivalent to a unique $f_{\hat{c}}$.

Construction is similar to Wittner capture and Timorin regluing. Based on

Thm.: (Thurston, Levy) A topological quadratic polynomial with periodic critical point is combinatorially equivalent to a unique polynomial.

6.2 Combinatorics and recapture

Thm.: Suppose $\hat{c}$ has the internal address 1-$\ldots$-k-n and c is the center of period qk before it, bifurcating from 1-$\ldots$-k. If $q = 2$, c may be the center of period k as well. Recall the preimages of 0, $z_j \in U_j$ with $f_c^{n-j}(z_j) = 0$.

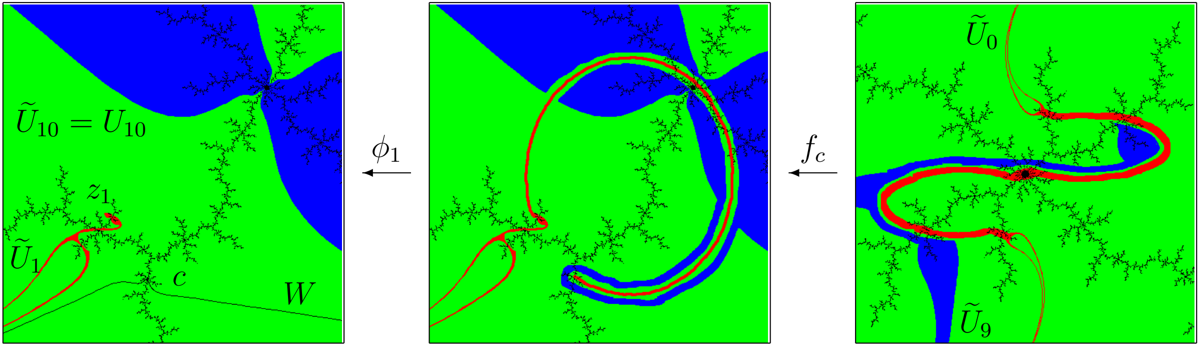
The arc γ from c to z_1 shall be homotopic to an arc disjoint from U_j for $j = 2, \ldots, n$. Then $\hat{c}$ is obtained from c by recapture along γ.

Proof: Choose $\Delta \subset W$ disjoint from U_j, $1 < j \le n$, and $g = \phi \circ f_c$. Then $g = f_c : U_j \to U_{j+1}$ for $1 \le j < n$. In $g^{-1}(U_1) \subset U_n$, choose an extended ray homotopic to its preimage. The quadratic polynomial equivalent to g will have extended rays with the same cyclic order landing on the critical orbit.

6.3 Recapture with early return

Recall the example of 1-2-6-7-13-14 with the early return $U_{10} \subset W$. Work in the dynamic plane of f_c with period 13 or 26.

If γ goes from c to z_1 directly, $g = \phi_0 \circ f_c$ will be equivalent to $f_{\hat{c}}$ for the center $\hat{c}$ with the correct internal address and the external angles $\frac{6347}{16383}$ and $\frac{6348}{16383}$.
If γ turns once around z_{10}, $g = \phi_1 \circ f_c$ will be equivalent to $f_{c'}$, where the center c' has the internal address 1-2-6-7-9-14 and the external angles $\frac{6443}{16383}$ and $\frac{6444}{16383}$.



6.4 More general curves:

Again, consider $\widehat{c}$ with the address 1-2-6-7-13-14, c' with 1-2-6-7-9-14, and c with 1-2-6-7-13 or 1-2-6-7-13-26 before them.

Let $\gamma_m \subset W$ be an arc from c to z_1 turning m times around z_{10}. Then $g = \phi_m \circ f_c$ will be equivalent to $f_{\widehat{c}}$ or to $f_{c'}$ depending on m.

Now $\phi_m \circ \phi_0^{-1}$ is isotopic to D^{-m} for a Dehn twist D about z_1 and z_{10}. Consider a Dehn twist $\widehat{D}$ about c_1 and c_{10}. Then $\widehat{D}^m \circ f_{\widehat{c}}$ will be equivalent to $f_{\widehat{c}}$ or to $f_{c'}$, respectively.

Compare this to the ...

6.5 Twisted Rabbit

Bartholdi–Nekrashevych have solved the Twisted Rabbit problem: f_c is the Rabbit polynomial and D is a Dehn twist about c_1 and c_2. Now consider $g = D^m \circ f_c$. Is it equivalent to the Airplane, the Rabbit, or its conjugate? All three types are realized for $m \in \mathbb{Z}$.

The images show that $g = D \circ f_c$ is equivalent to the Airplane, since the curves representing extended rays are homotopic to their preimages; the circular order determines the polynomial.