Self-similarity of the Mandelbrot set

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The images are made with Mandel, a program available from www.mndynamics.com .

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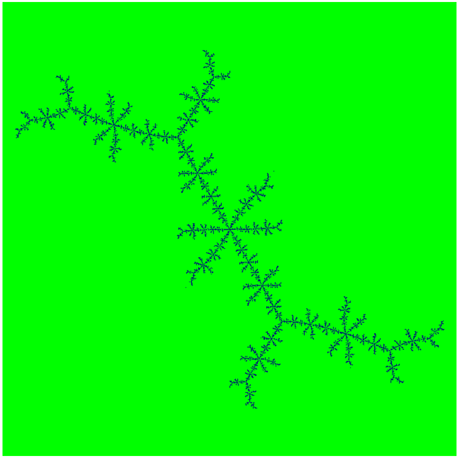
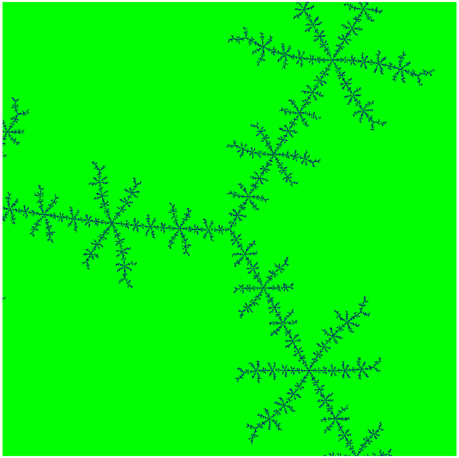
1a Dynamics and Julia sets

Iteration of $f_c(z) = z^2 + c$. Filled Julia set $\mathcal{K}_c = \{z \in \mathbb{C} \mid f_c^n(z) \not\to \infty\}$.

$\mathcal{K}_c$ is invariant under $f_c(z)$.

$f_c^n(z)$ is asymptotically linear at repelling n-periodic points.



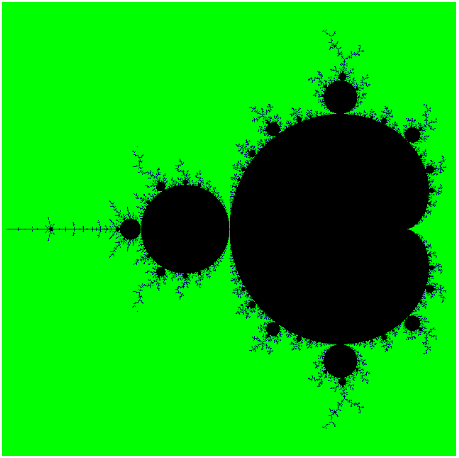


1b The Mandelbrot set

Parameter plane of quadratic polynomials $f_c(z) = z^2 + c$.
Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} \,|\, c \in \mathcal{K}_c\}$.

Two kinds of self-similarity:

- Convergence of rescaled subsets: geometry is asymptotically linear.
- Homeomorphisms between subsets: non-linear, maybe quasiconformal.



2. Asymptotic similarity

Subsets $\mathcal{M} \supset \mathcal{M}_n \to \{a\} \subset \partial\mathcal{M}$ with asymptotic model: $\varphi_n(\mathcal{M}_n) \to Y$ in Hausdorff-Chabauty distance.

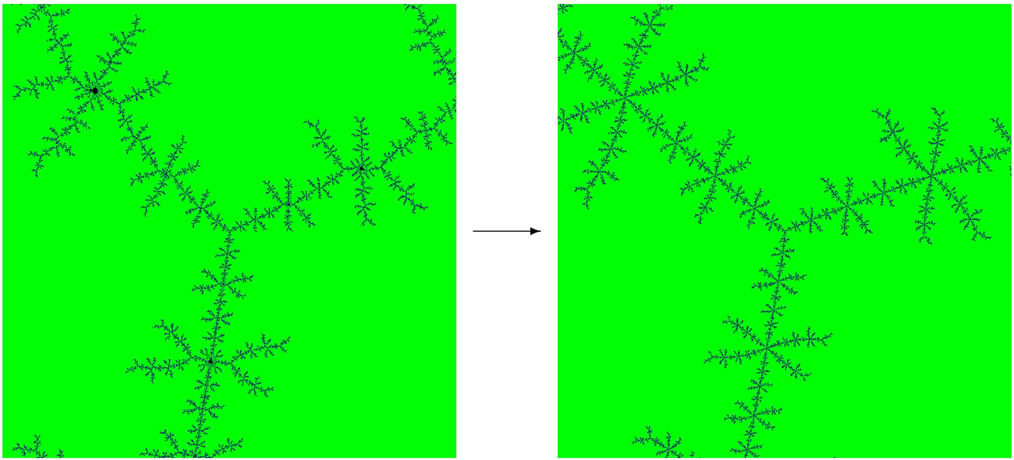
Or convergence of subsets of different Julia sets: $c_n \in \mathcal{M}_n$, $\mathcal{K}_n \subset \mathcal{K}_{c_n}$ with asymptotics $\psi_n(\mathcal{K}_n) \to Z$.

Maybe $Z = \lambda Y$.

2a Misiurewicz points

Misiurewicz point a with multiplier ϱ_a. Blowing up both planes with ϱ_a^n gives $\lambda(\mathcal{M} - a) \approx (\mathcal{K}_a - a)$ asymptotically.

Classical result by Tan Lei. Error bound by Rivera-Letelier. Generalization to non-hyperbolic semi-hyperbolic a. Proof by Kawahira with Zalcman Lemma.



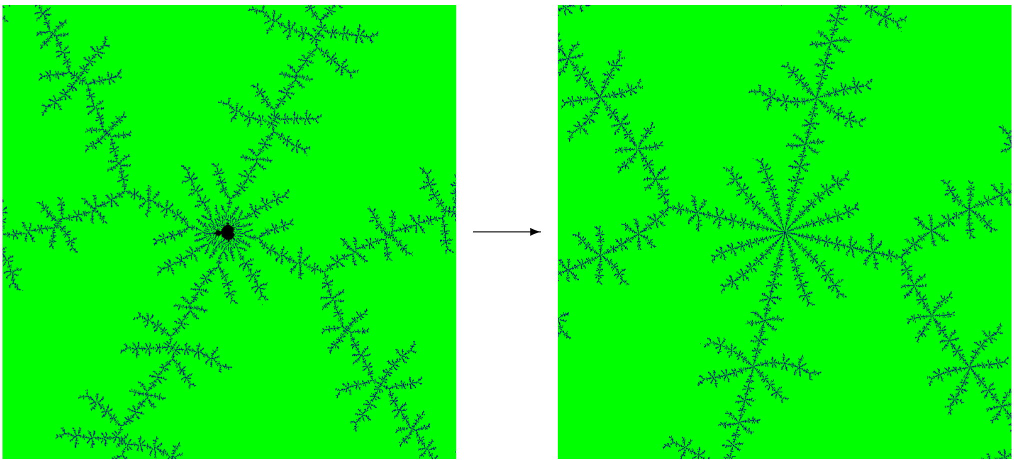
2b Multiple scales (1)

Centers $c_n \to a$ with $c_n \sim a + K\varrho_a^{-n}$ and small Mandelbrot sets of diameter $\asymp |\varrho_a|^{-2n}$. Their decorations have asymptotic models on intermediate scales:

$$\varrho_a^{1n}(\mathcal{M} - c_n) \to K_0(X_a - \varphi_a(0))^1$$

$$\varrho_a^{\frac{3}{2}n}(\mathcal{M} - c_n) \to K_1(X_a - \varphi_a(0))^{1/2}$$

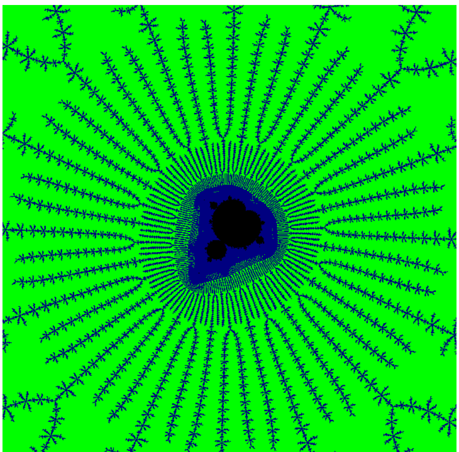
$$\varrho_a^{\frac{7}{4}n}(\mathcal{M} - c_n) \to K_2(X_a - \varphi_a(0))^{1/4}$$

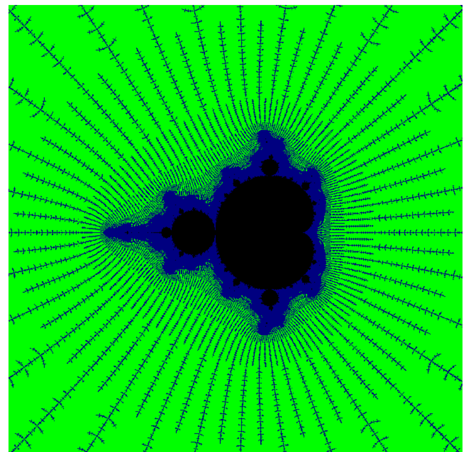


2b Multiple scales (2)

Consider decorations on the smallest scale ϱ_a^{-2n}.
Then we have the following non-hairiness properties:

- For a fixed dyadic internal angle and $n \to \infty$, the decoration is close to an analytic arc. This arc is a small dyadic ray for $a \in \mathbb{R}$.
- The union of decorations gets dense. Area $\to 0$? [Proved for $a = -2$.]



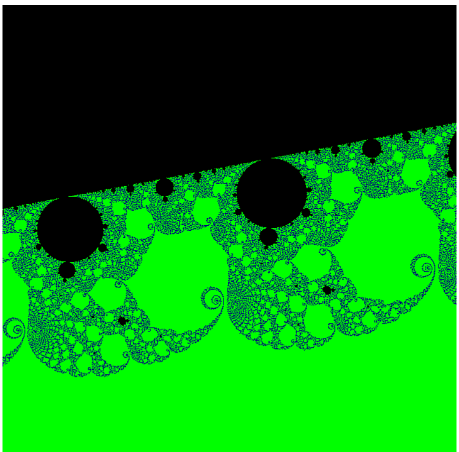


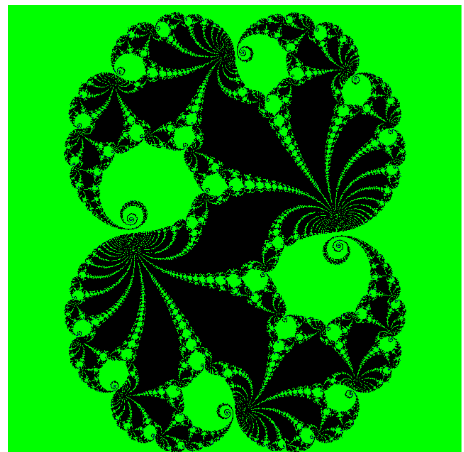
2c The Fibonacci parameter

A real parameter with fast recurrent critical point. According to Lyubich and Wenstrom, suitable puzzle-pieces approximate the basilica.

2d Elephant and dragon

Rescaled limbs of $\mathcal{M}$ seem to converge to a limit set. Partial proof by Lavaurs–Douady based on parabolic implosion. Mandel demo 2.10

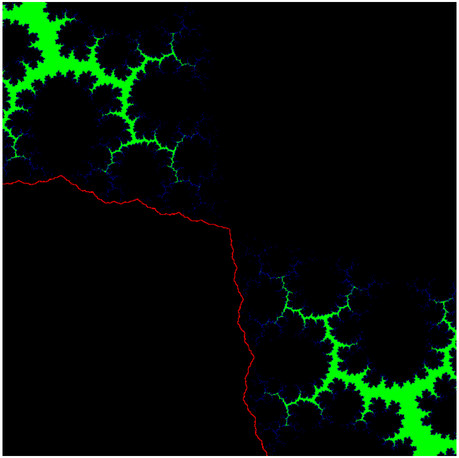




2e Siegel parameters (1)

The Siegel disk for the Golden Mean rotation number. According to McMullen, the filled Julia set is asymptotically linearly self-similar at the critical point.

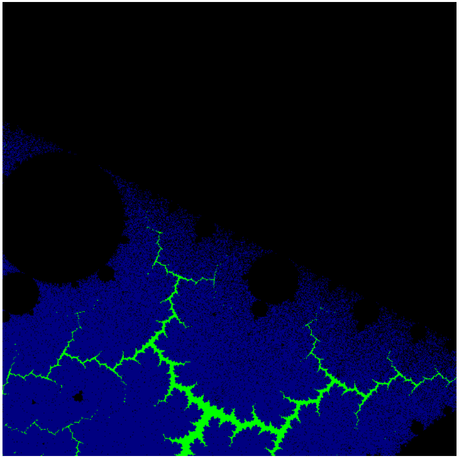
It is a Lebesgue full density point of the filled Julia set. According to Buff and Henriksen, there is a triangle at 0 within the Siegel disk.

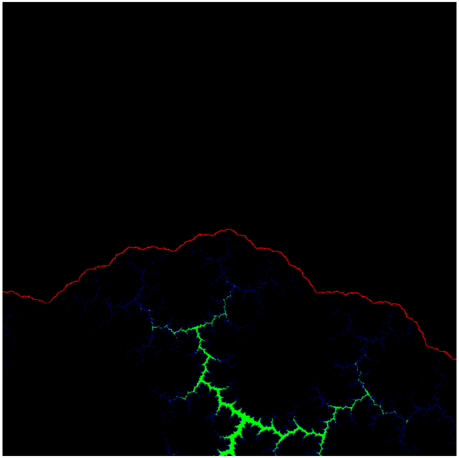


2e Siegel parameters (2)

The right image shows the self-similarity of the filled Julia set at the critical value.

Is there an approximate self-similarity in the parameter plane as well, and a similarity between both planes?



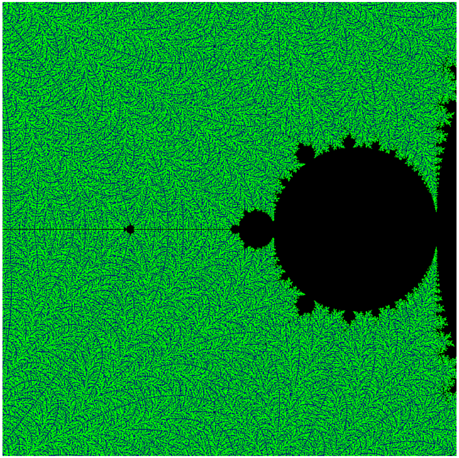


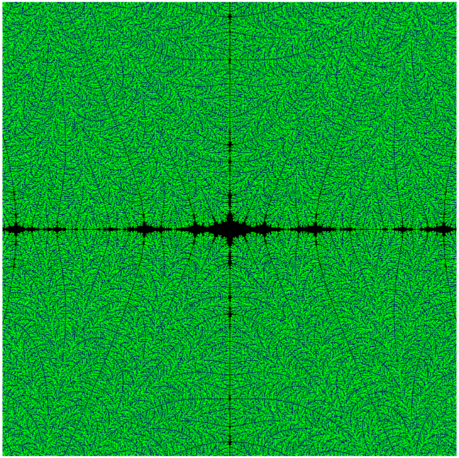
2f Feigenbaum doubling

The Feigenbaum parameter $c_F \approx -1.40115519$ is the limit of real period doubling. Consider $\delta_F^n(\mathcal{M} - c_F)$ with $\delta_F \approx 4.66920161$.

Milnor had conjectured that the hairs converge in measure to a limit set, which is sparse somewhere. Lyubich has shown that hairs become dense.

The Julia set was discussed by McMullen. Mandel demo 5.10

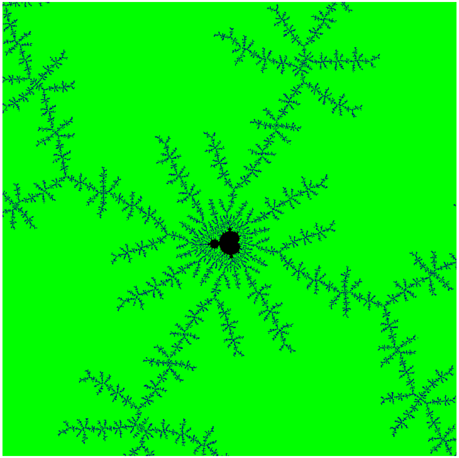


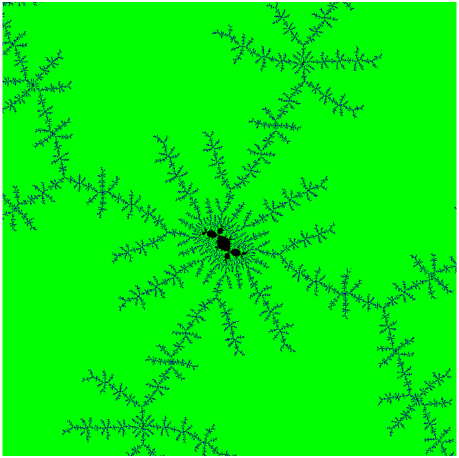


2g Local similarity

Similarity between the decorations of small Mandelbrot sets and of corresponding small Julia sets. Composition of small Douady map and Boettcher maps; approximately affine on a larger radius.

Observed by Peitgen 1988. Proof at Misiurewicz points using large modulus, bounded geometry, and asymptotic models. Mandel demo 7.2

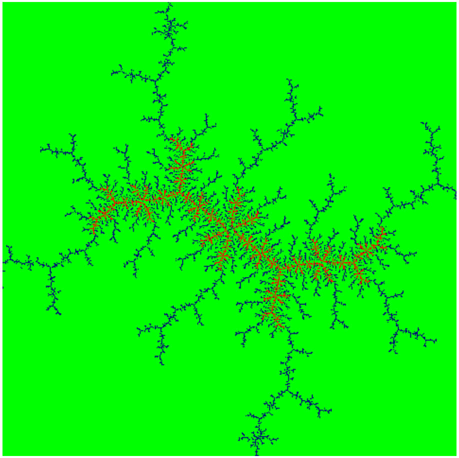


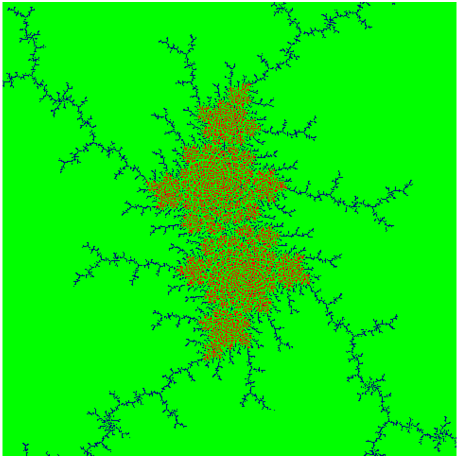


2h Embedded Julia sets

The boundary of a primitive small Mandelbrot set is accumulated by embedded Julia sets. These Cantor subsets of $\mathcal{M}$ correspond to preimages of the disconnected small Julia set.

Observed by Leavitt–Munafo. Discussed by Douady–alii at parabolic parameters. Otherwise convergent.





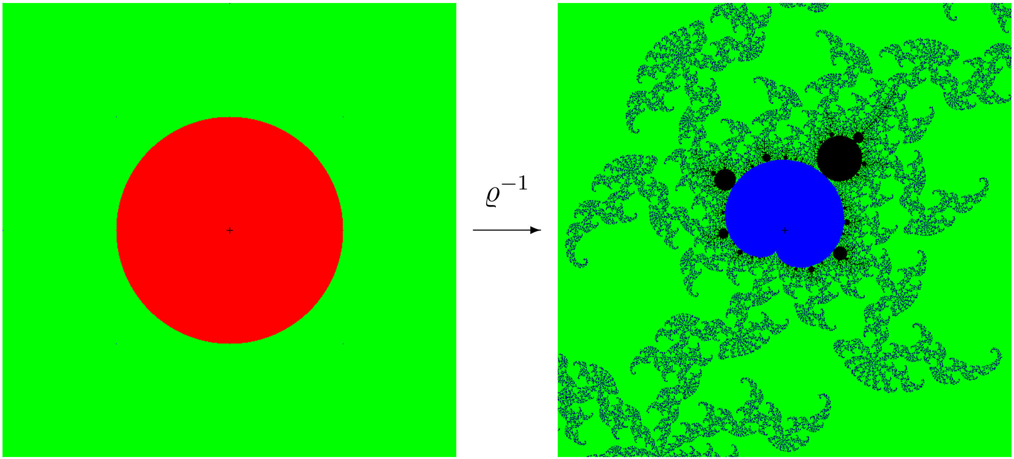
Quasiconformal surgery has many applications to single maps. E.g., Shishikura constructed Herman rings from Siegel disks.

Applied to families of maps, it can yield a homeomorphism between parameter spaces:

- Using $f_c(z)$, construct a quasiregular $g_c(z)$ piecewise.
- Conjugate $g_c(z)$ to an analytic $f_{\widehat{c}}(z)$.
- Map parameters according to $c \mapsto h(c) := \widehat{c}$.

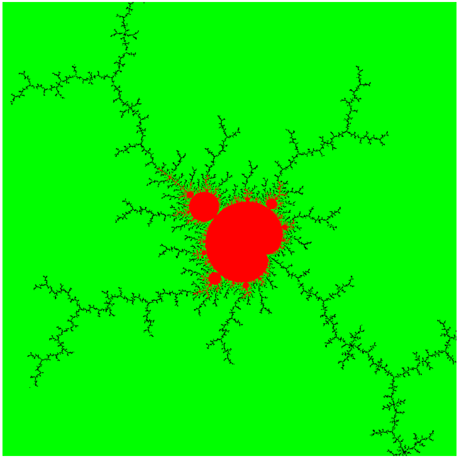
3a Multiplier map

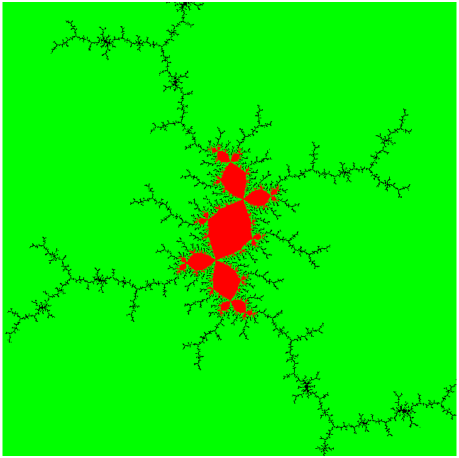
For a hyperbolic component $\Omega \subset \mathcal{M}$, the multiplier map $\varrho : \Omega \to \mathbb{D}$ is analytic. To show that it is injective, a continuous ϱ^{-1} is constructed by surgery.



3b Simple renormalization

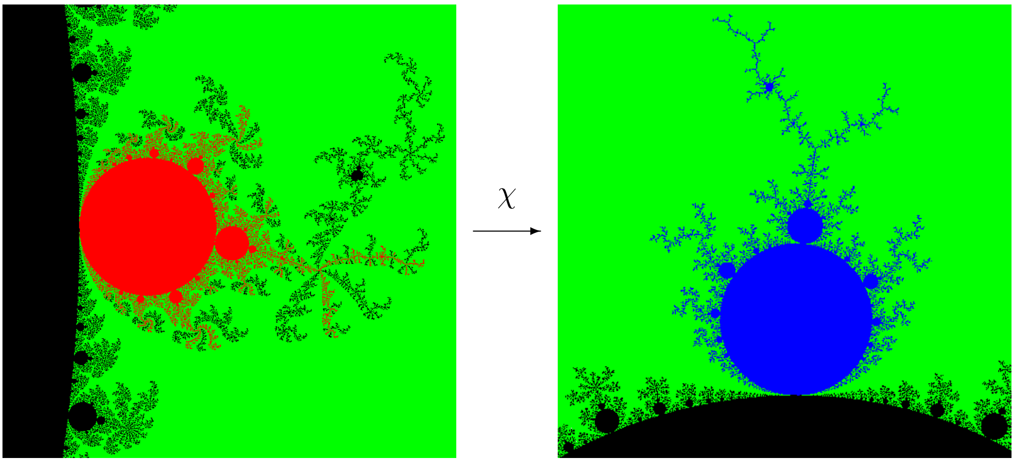
A small Julia set $\mathcal{K}_c^n \subset \mathcal{K}_c$ is obtained from a quadratic-like restriction $g_c(z)$ of $f_c^n(z)$. When $\mathcal{K}_c^n$ is connected, then c belongs to the small Mandelbrot set $\mathcal{M}_n$. Now $\chi : \mathcal{M}_n \to \mathcal{M}$ is a homeomorphism. $\mathcal{M} \setminus \mathcal{M}_n$ and $\mathcal{K}_c \setminus \mathcal{K}_c^n$ is a countable family of decorations.





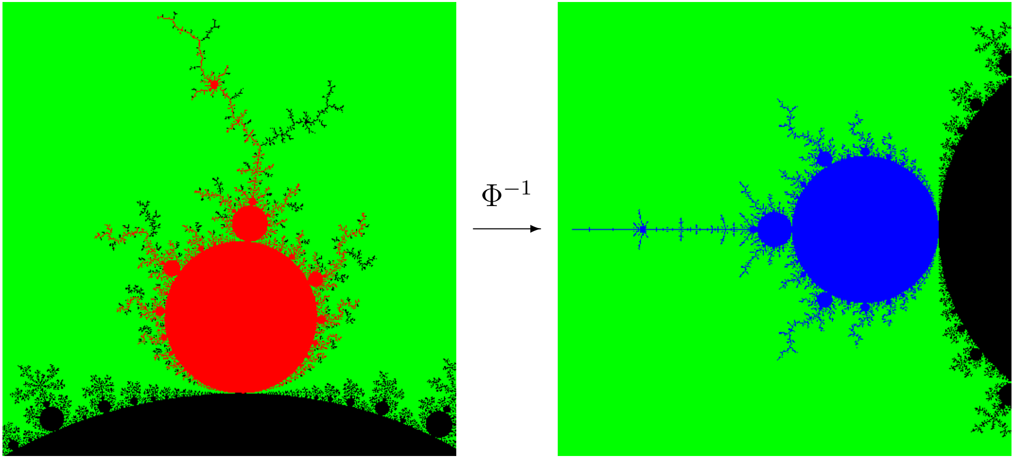
3c Crossed renormalization

Crossed renormalization was discovered by McMullen and described by Riedl–Schleicher. The example shows a subset of the $1/6$-limb, which is homeomorphic to the $1/3$-limb.



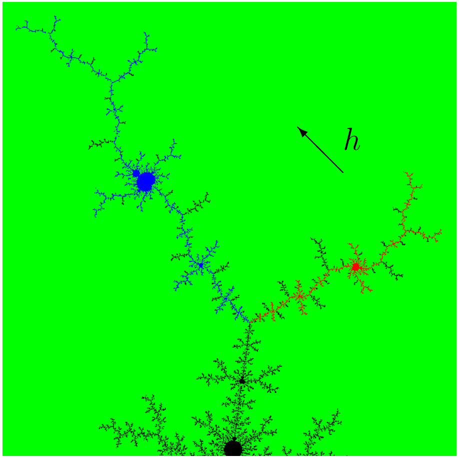
3d Branner–Douady

This homeomorphism provides an embedding of the $1/2$-limb into the $1/3$-limb. The image of the real interval is the vein to a β-type Misiurewicz point. The construction of Φ^{-1} is easier than that of Φ.

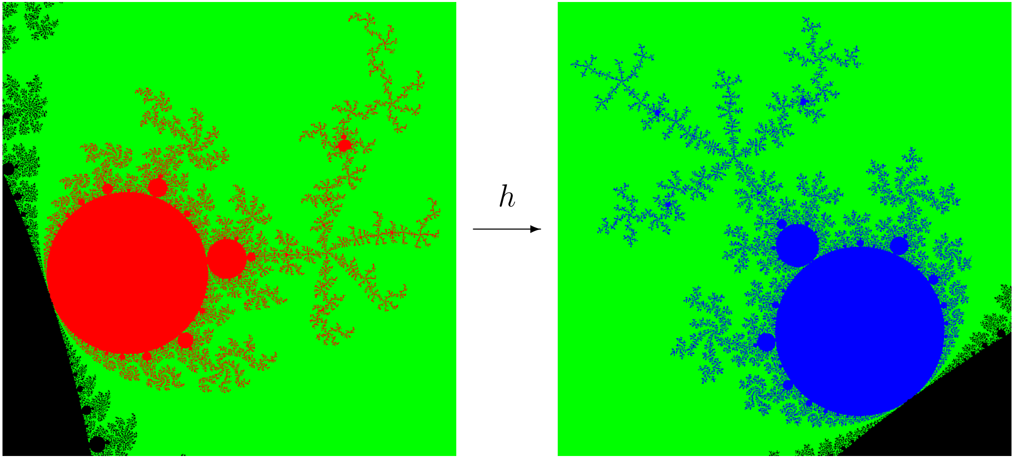


3e Riedl

Riedl has obtained all veins by applying two kinds of homeomorphisms iteratively: these are embedding sublimbs of a hyperbolic component, or interchanging subsets of branches behind a Misiurewicz point.



3f Branner-Fagella and Schleicher have constructed homeomorphisms between full limbs of equal denominators.

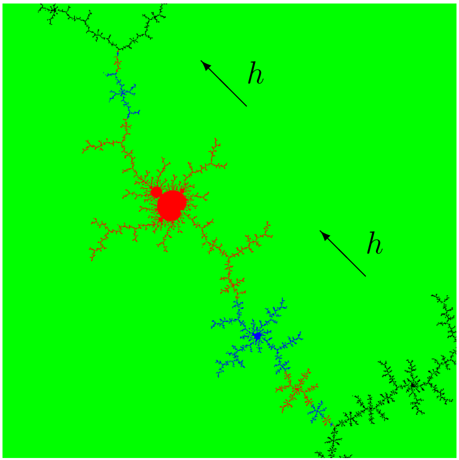


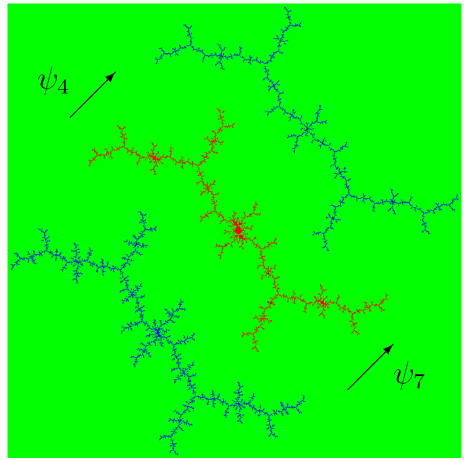
3g Non-orientation-preserving: composition with complex conjugation.

Dudko-Schleicher have obtained homeomorphisms between limbs or sublimbs, which are orientation-preserving except at α-type Misiurewicz points, e.g.. They are constructed combinatorially; continuity follows from the Yoccoz Theorem and the Decoration Theorem.

3h Homeomorphisms on edges

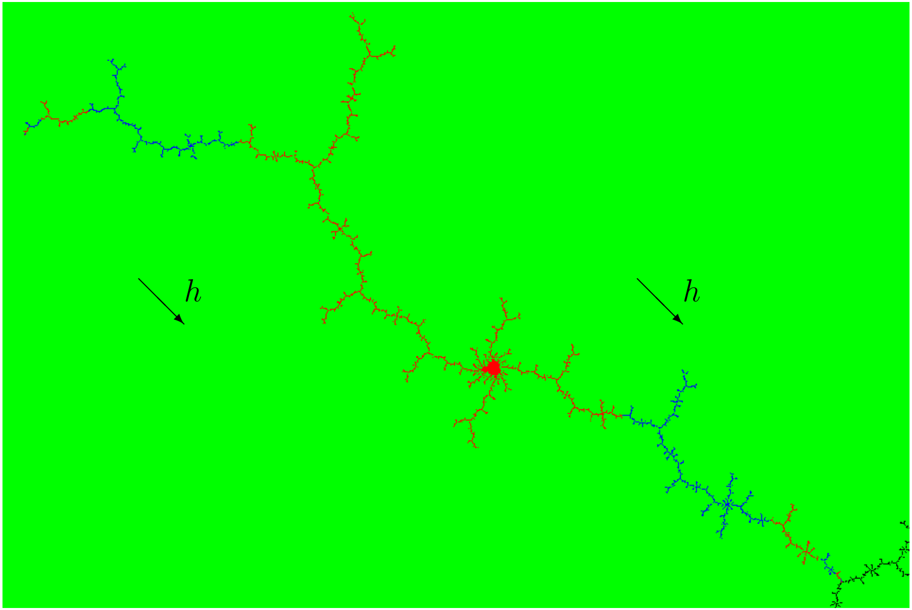
These homeomorphisms are mapping the part between two vertices to itself. There is a countable family of mutually homeomorphic subsets. No decorations are excluded.





3i Homeomorphisms at endpoints

The same construction where one vertex is an endpoint. At both Misiurewicz points, fundamental domains are defined combinatorially, are mutually homeomorphic, and scale asymptotically linearly. Now h is Lipschitz at these vertices, but not differentiable.



Combinatorial assumptions:

- Subsets $\mathcal{E}_{\scriptscriptstyle M} \subset \mathcal{M}$ and corresponding $\mathcal{E}_c \subset \mathcal{K}_c$.
- By composition of iterates of $f_c^{\pm 1}(z)$, a dynamic homeomorphism $\eta_c : \mathcal{E}_c \to \mathcal{E}_c$ is defined piecewise. The pieces are bounded by stable preperiodic rays.

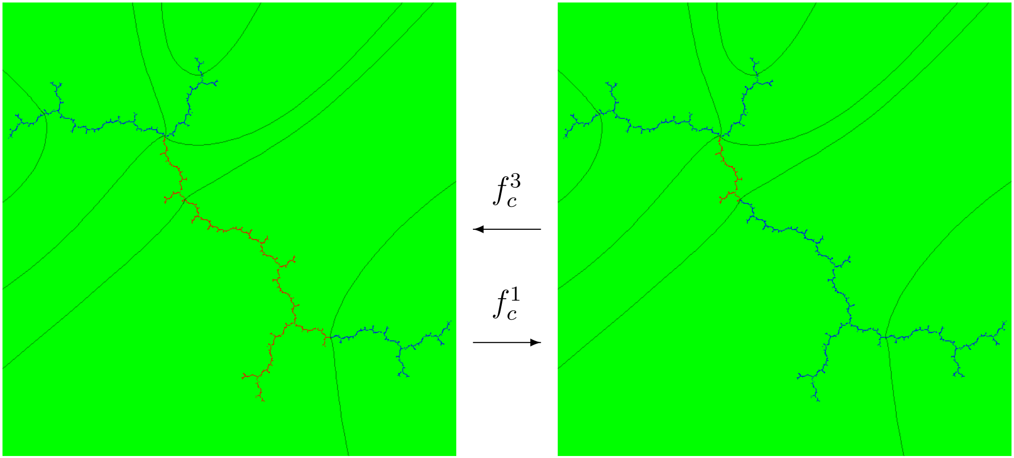
Theorem: $g_c \sim f_c \circ \eta_c$ is straightened to some $f_{\widehat{c}}$ and $h(c) := \widehat{c}$ defines a homeomorphism $h : \mathcal{E}_{\scriptscriptstyle M} \to \mathcal{E}_{\scriptscriptstyle M}$.

Corollary: The group of non-trivial orientation-preserving homeomorphisms of $\mathcal{M}$ has the cardinality of $\mathbb{N}^{\mathbb{N}} \sim \mathbb{R}$.

4a Piecewise dynamic homeomorphisms

In Example 3i, $f_c^4(\mathcal{E}_c)$ is the largest branch at α_c, which contains 0 and β_c. This branch is mapped to itself piecewise by f_c^1 and f_c^{-3}.

So $\eta_c : \mathcal{E}_c \to \mathcal{E}_c$ is defined piecewise by $f_c^{-4} \circ f_c^1 \circ f_c^4$ and $f_c^{-4} \circ f_c^{-3} \circ f_c^4$. Extend it by the identity and consider $g_c \sim f_c \circ \eta_c$.



4b Combinatorial surgery

The map $f_c \circ \eta_c$ has shift discontinuities on four (or six) dynamic rays, but the new dynamics on $\mathcal{K}_c$ is well-defined. In the postcritically finite case, $\widehat{c}$ is obtained from c by determining its Hubbard tree or its external angles.

Or a Hölder homeomorphisms of angles is determined, conjugating some piecewise linear map to the angle doubling map. From the map of angles, h could be defined on $\mathcal{M}$ without assuming local connectivity, by employing the Yoccoz Theorem and the Decoration Theorem.

4c Straightening a quasi-quadratic map

$f_c \circ \eta_c$ is modified in four (or six) sectors and restricted to a bounded domain to obtain a quasi-regular quadratic-like map $g_c : U'_c \to U_c$.

Since the sectors are preperiodic, all iterates g_c^n have uniformly bounded dilatation.

Adapting the proof of the Straightening Theorem, there is a hybrid equivalence ψ_c conjugating g_c to a quadratic polynomial $f_{\widehat{c}}$.

4d The homeomorphism

Continuity of $h(c) = \widehat{c}$ is shown from:

- An explicit representation in the exterior and interior of $\mathcal{M}$.
- Quasi-conformal rigidity at $\partial\mathcal{M}$.

The techniques are similar to renormalization but easier in fact, since the holomorphic motion of $U_c \setminus U_c'$ is given explicitly by the composition of Boettcher conjugations. h^{-1} is constructed analogously.

Thank you.