Quadratic matings and anti-matings

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Abstract

. . . based on Ahmadi Dastjerdi, Timorin, Meyer.

1 Introduction

complex dynamics means iteration of polynomials and rational maps

related by matings and captures, mating glues quadratic Julia sets

equator and anti-equator observed

no f : Kp to Kq

blaschke? poly? idea two planes, gives quartic Julia sets. notation

here construction based on gluing quartic Julia sets mapped to each other

parallel description mating and anti-mating to emphasize similar notions and techniques

mention formal essential topological geometric combinatorial, alternatively with laminations

remark notation does not mean intersection

in Sections: pcf Thurston, results on existence and also equators: but preper more general

several examples and discussion of special families

naming and history: called cross-mating in an earlier version of [40], which has the advantage of no confusion with anti-holomorphic matings.

[31, 1], lamination Ahmadi Dastjerdi, anti-eq Meyer [4, 24], Timorin laminations and blaschke and regluing cf encapture, neither toplogical, straightening Inou–Kiwi.

So here the existence result is the same as [1], with formal mating instead of laminations. New: glue quartics, Lattès, many examples and special families, Hausdorff obstruction, numerics, equator.

While the simplest mating is $f(z) = z^2$, the prototypical anti-mating is given by $f(z) = z^{-2}$. Anti-mating is understood by gluing two filled Julia sets, which are mapped to each other by f. The notion of an anti-equator and pseudo-anti-equator was suggested by Daniel Meyer [4, 24]. Vladlen Timorin has defined the formal anti-mating, and suggested a definition in terms of laminations in addition [40]. This definition goes back to the thesis [1] of Davood Ahmadi Dastjerdi.

The basic idea is that the formal anti-mating $g = P \sqcap Q$ and the topological anti-mating $P \prod Q$ are quadratic maps, but the topology and dynamics is understood from the quartic polynomials $Q \circ P$ and $P \circ Q$. Except in the symmetric case of $P = Q$, the

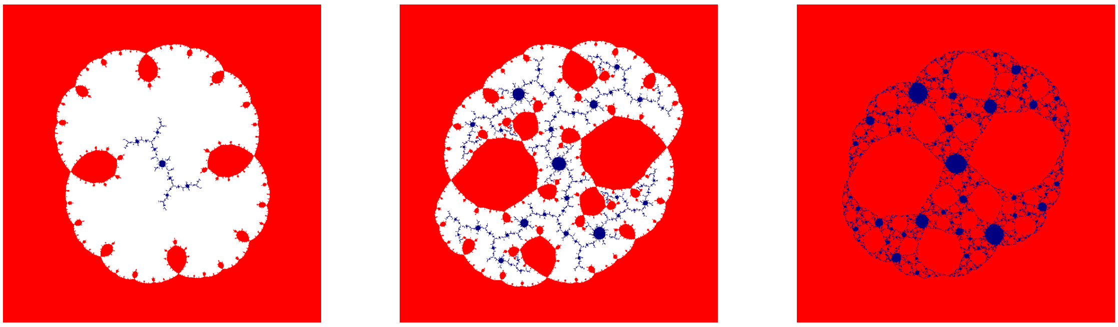


Figure 1: The mating of Kokopelli and Basilica gives a quadratic rational map f, such that $z = 0$ is 4-periodic and $z = \infty$ is 2-periodic. Left: formal mating, middle: Thurston iteration, right: geometric mating. The same map f is obtained as an anti-mating in Figure 2. See also Example 2.2.

quadratic dynamics of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ will be irrelevant. It turns out that the anti-matings with $p = 0$ belong to the bitransitive family, while $p = -q^2$ gives the critically 2-periodic family V_2 [40]. See also the videos on www.mndynamics.com .

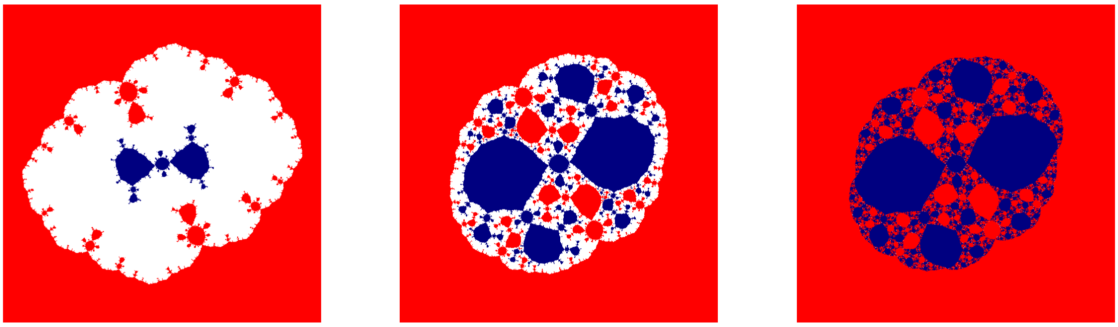


Figure 2: An anti-mating giving the same quadratic rational map f as in Figure 1, $z = 0$ is 4-periodic and $z = \infty$ is 2-periodic. Since the two quartic polynomials are semi-conjugate, the Julia sets show similar features: each seems to contain parts of the other. The 4-periodic basins appear more round and the 2-periodic basins show prominent cusps. Left: formal anti-mating, middle: Thurston iteration, right: geometric anti-mating. See Figure 5 for the parameter spaces.

Acknowledgment: Several colleagues have contributed to this work by inspiring discussions and helpful suggestions. I wish to thank in particular Arnaud Chéritat, Sarah Koch, Kathryn Lindsey, Daniel Meyer, Carsten Petersen, Pascale Roesch, Tom Sharland, and Vladlen Timorin. And special thanks to Mary Rees and Lasse Rempe for providing a copy of [1] on short notice.

2 Definitions and basic properties

Formal matings and anti-matings, as well as corresponding rational maps, are defined and discussed in Sections 2.2–2.5. While the formal (anti-)mating is never conjugate to a rational map, it forms a good starting point for the construction of rational maps with desired dynamic or topological properties: the combinatorial and geometric (anti-)matings. An implementation of the Thurston algorithm [7] is described briefly in Section 2.6.

2.1 Polynomial and rational dynamics

f and K and M, limb closed by adding root, Basilica Airplane Rabbit Kokopelli, Boettcher, unit disk $\mathbb{D}$, rays, doubling, pinching points and branch points, M limbs and what for higher degree?, rational two crpt, branch portrait and types hyperbolic in Section 2.5

2.2 Formal mating and anti-mating

The formal mating or formal anti-mating is a map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere, which combines the dynamics of polynomials $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$. To identify the two planes with hemispheres, the homeomorphisms $\varphi_0 : \mathbb{C} \to \mathbb{D}$ and $\varphi_\infty : \mathbb{C} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ shall be used; any map with the same asymptotics at ∞ would do:

$$\varphi(z) = \frac{z}{\sqrt{|z|^2 + 1}} \qquad \text{and} \qquad \varphi_\infty(z) = 1/\varphi_0(z) \ . \tag{1}$$

Definition 2.1 (Formal mating and anti-mating)
The formal mating or formal anti-mating $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of two quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ is defined on the southern and northern hemispheres in terms of $\varphi_0 : \mathbb{C} \to \mathbb{D}$ and $\varphi_\infty : \mathbb{C} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ according to (1):
1. *If the quadratic Julia sets $\mathcal{K}_P$ and $\mathcal{K}_Q$ are locally connected, define the formal mating $g = P \sqcup Q$ as*

$$P \sqcup Q(z) = \begin{cases} \varphi_0 \circ P \circ \varphi_0^{-1}(z) & , \quad z \in \mathbb{D} \\ z^2 & , \quad z \in \partial\mathbb{D} \\ \varphi_\infty \circ Q \circ \varphi_\infty^{-1}(z) & , \quad z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \end{cases} \tag{2}$$

For an angle $\theta \in S^1$, the external ray of g is defined as

$$\mathcal{R}_g(\theta) = \varphi_0(\mathcal{R}_P(\theta)) \cup \{\exp(\mathrm{i}2\pi\theta)\} \cup \varphi_\infty(\mathcal{R}_Q(-\theta)) \ . \tag{3}$$

Now g maps $\mathcal{R}_g(\theta)$ to $\mathcal{R}_g(2\theta)$.
2. *If the quartic Julia sets $\mathcal{K}_{QP}$ and thus $\mathcal{K}_{PQ}$ are locally connected, define the formal anti-mating $g = P \sqcap Q$ as*

$$P \sqcap Q(z) = \begin{cases} \varphi_\infty \circ P \circ \varphi_0^{-1}(z) & , \quad z \in \mathbb{D} \\ z^{-2} & , \quad z \in \partial\mathbb{D} \\ \varphi_0 \circ Q \circ \varphi_\infty^{-1}(z) & , \quad z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \end{cases} \tag{4}$$

For an angle $\theta \in S^1$, the external ray of g is defined as

$$\mathcal{R}_g(\theta) = \varphi_0(\mathcal{R}_{QP}(\theta)) \cup \{\exp(\mathrm{i}2\pi\theta)\} \cup \varphi_\infty(\mathcal{R}_{PQ}(-\theta)) \ . \tag{5}$$

So g maps $\mathcal{R}_g(\theta)$ to $\mathcal{R}_g(-2\theta)$, reversing its direction relative to the equator $\partial\mathbb{D}$.

Now $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a smooth branched cover with critical points 0 and ∞, and a distinguished fixed point at 1. Although the same notation of $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ is used in both cases, the dynamic meaning for g^n is quite different:

- In the context of a mating, the quadratic dynamics of P^n and Q^n is relevant. $g = P \sqcup Q$ maps each of the quadratic Julia sets $\varphi_0(\mathcal{K}_P)$ and $\varphi_\infty(\mathcal{K}_Q)$ to itself.

- For the anti-mating $g = P \sqcap Q$, the quadratic dynamics of P^n and Q^n is irrelevant. g maps each of the quartic Julia sets $\varphi_0(\mathcal{K}_{QP})$ and $\varphi_\infty(\mathcal{K}_{PQ})$ to the other one. Note also that the iterate $(P \sqcap Q)^2 = (Q \circ P) \sqcup (P \circ Q)$ is a quartic mating.

In the annulus between the filled Julia sets, every orbit escapes to the equator. The dynamics of the anti-mating may be postcritically finite, e.g., if p and q are not within the Mandelbrot set; even if they are, the dynamics of $Q \circ P$ and $P \circ Q$ will be unrelated to that of P and Q in general. The associativity of composition,

$$P \circ (Q \circ P) = (P \circ Q) \circ P \qquad \text{and} \qquad Q \circ (P \circ Q) = (Q \circ P) \circ Q \ , \tag{6}$$

means that the quartic polynomials are semi-conjugate in both directions; for this reason we have $P : \mathcal{K}_{QP} \to \mathcal{K}_{PQ}$ and $Q : \mathcal{K}_{PQ} \to \mathcal{K}_{QP}$ [1]. Moreover, these quadratic polynomials are conjugate to $z \mapsto z^2$ in the exterior, doubling the angles of quartic external rays according to $P : \mathcal{R}_{QP}(\theta) \to \mathcal{R}_{PQ}(2\theta)$ and $Q : \mathcal{R}_{PQ}(\theta) \to \mathcal{R}_{QP}(2\theta)$. — These concepts are explained in greater detail in the following Section 2.3:

2.3 Dynamics and normalizations of the formal anti-mating

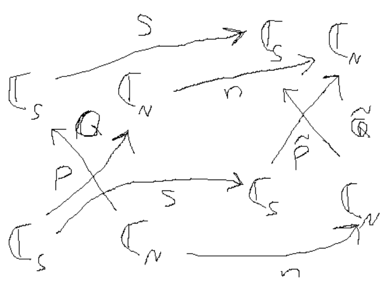
A formal mating $P \sqcup Q$ maps each hemisphere to itself, so on a first glance, we may choose two kinds of dynamics independently of each other. On a second glance, ray connections must be taken into account, which determine whether the geometric mating exists, and how the dynamics is modified by identifications.

For a formal anti-mating $g = P \sqcap Q$, already the first step is more involved. The dynamics of g is understood from the following toy model $\widehat{g}$, which is conjugate to g on $\widehat{\mathbb{C}} \setminus \partial\mathbb{D}$:

- Consider the disjoint union $\mathbb{C}_S \uplus \mathbb{C}_N$ of two copies of $\mathbb{C}$.

- Given two quadratic polynomials $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$, define a self-map $\widehat{g}$ of $\mathbb{C}_S \uplus \mathbb{C}_N$ by $P : \mathbb{C}_S \to \mathbb{C}_N$ and $Q : \mathbb{C}_N \to \mathbb{C}_S$.

- The dynamics of $\widehat{g}$ may be conjugated with the disjoint union of two affine maps, $s : \mathbb{C}_S \to \mathbb{C}_S$ and $n : \mathbb{C}_N \to \mathbb{C}_N$. This gives new quadratic polynomials $\widetilde{P} = n \circ P \circ s^{-1}$ and $\widetilde{Q} = s \circ Q \circ n^{-1}$ according to the commuting diagrams:



(7)

- The normal form $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ is obtained by appropriate affine conjugations in the sense of this diagram: the critical points are shifted to 0, and then polynomials can be made monic by suitable rescalings. The formulas show that there are three pairs of monic polynomials, such that the toy models $\widehat{g}$ are affine conjugate: with ζ a cubic root of unity, we may replace $p \to \zeta p$ and $q \to \zeta^{-1} q$. Then the formal anti-mating g will be rotated with ζ as well; the combinatorial and geometric anti-matings according to Section 2.4 will be rescaled linearly, such that another fixed point is normalized to 1.

Now we want to study the dynamics of $\widehat{g}$, the behavior of points in $\mathbb{C}_S \uplus \mathbb{C}_N$ under iteration $\widehat{g}^n$. If, e.g., $z \in \mathbb{C}_S$, the sequence of iterates $\widehat{g}^n(z)$ alternates between both planes as follows: $\widehat{g}^{2k}(z) = (Q \circ P)^k(z) \in \mathbb{C}_S$ and $\widehat{g}^{2k+1}(z) = P \circ (Q \circ P)^k(z) \in \mathbb{C}_N$. And vice versa for $z \in \mathbb{C}_N$; see also (9). Notions like periodic points, critical points, postcritical finiteness, escaping set, and filled Julia set of $\widehat{g}$ are understood with respect to this alternating kind of iteration. — Recall the semi-conjugations

$$P \circ (Q \circ P)^n = (P \circ Q)^n \circ P \qquad \text{and} \qquad Q \circ (P \circ Q)^n = (Q \circ P)^n \circ Q\ . \qquad (8)$$

- $\widehat{g}$ has two critical points, $0 \in \mathbb{C}_S$ and $0 \in \mathbb{C}_N$. The two critical orbits need not be disjoint.

- The filled Julia set $\mathcal{K}_{\widehat{g}}$ is the disjoint union of $\mathcal{K}_{QP} \subset \mathbb{C}_S$ and $\mathcal{K}_{PQ} \subset \mathbb{C}_N$. By the semi-conjugations in (6) or (8) we have $P : \mathcal{K}_{QP} \to \mathcal{K}_{PQ}$ and $Q : \mathcal{K}_{PQ} \to \mathcal{K}_{QP}$, so $\widehat{g} : \mathcal{K}_{\widehat{g}} \to \mathcal{K}_{\widehat{g}}$. This is Lemma 5.14.4 in [1].

- Now $\mathcal{K}_{QP}$ and $\mathcal{K}_{PQ}$ are connected, if and only if both critical orbits of $\widehat{g}$ are bounded. They are locally connected especially, if the critical orbits are finite. — These statements correspond to standard results on polynomial dynamics [28]. Note here that $Q \circ P$ has two critical orbits, $(Q \circ P)^n(0)$ and $(Q \circ P)^n(\pm\sqrt{-p})$, but the latter is just $Q \circ (P \circ Q)^{n-1}(0)$.

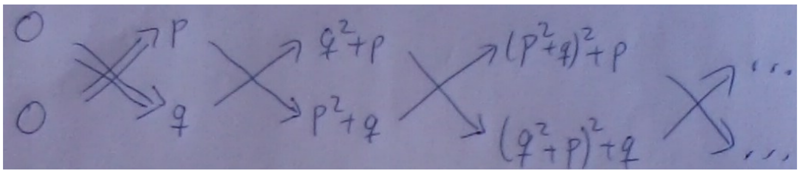
- Then we have the quartic Boettcher maps $\Phi_{QP} : \widehat{\mathbb{C}}_S \setminus \mathcal{K}_{QP} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\Phi_{PQ} : \widehat{\mathbb{C}}_N \setminus \mathcal{K}_{PQ} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, which conjugate the quartic polynomials to $z \mapsto z^4$. These maps are determined uniquely from the condition $\Phi_*(z) \sim z$ as $z \to \infty$. External rays are preimages of straight rays under the respective Boettcher conjugation.

- Now $\Phi_{PQ} \circ P \circ \Phi_{QP}^{-1} : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ equals $z \mapsto z^2$, since it is a Blaschke product and it is asymptotic to z^2 at ∞. Therefore P maps rays to rays, doubling the angle: $\mathcal{R}_{QP}(\theta) \mapsto \mathcal{R}_{PQ}(2\theta)$, and analogously for Q.

- Angles are complex conjugated, when $\mathbb{C}_N$ is mapped to the northern hemisphere with φ_∞, so the external rays $\mathcal{R}_g(\theta)$ of g according to Definition 2.1 are connected arcs. They are represented in the toy model by $\mathcal{R}_{\widehat{g}}(\theta) = \mathcal{R}_{QP}(\theta) \uplus \mathcal{R}_{PQ}(-\theta)$. Now $\widehat{g}$ maps this ray to $Q(\mathcal{R}_{PQ}(-\theta)) \uplus P(\mathcal{R}_{QP}(\theta)) = \mathcal{R}_{QP}(-2\theta) \uplus \mathcal{R}_{PQ}(2\theta) = \mathcal{R}_{\widehat{g}}(-2\theta)$.

The normalization of P and Q serves not only to simplify the formulas of the polynomials and the normalization of the Boettcher conjugation — it determines implicitly, how the two hemispheres connect to form a sphere. Let us compare this to the mating of two arbitrary quartic polynomials: for each of these, there are three choices of a Boettcher conjugation, or three choices, which of the fixed rays has angle $\theta = 0$. So there are nine different formal matings, and three different matings up to a common rotation: the formal mating is rotated when one polynomial is rotated with ζ and the other one with ζ^{-1}. Other kinds of rotations give different matings, which may be described by Dehn twists about the equator as well [3]. Returning to our quadratic anti-matings, the three rays with angles $\theta = 0$, $1/3$, $2/3$ will be fixed by the formal anti-mating. g is conjugated with a rotation by replacing $p \to \zeta p$ and $q \to \zeta^{-1} q$ at the same time, with ζ a cubic root of unity. Any other rotation would change the dynamics randomly, maybe producing disconnected Julia sets.

The alternating critical orbits of $\widehat{g}$ may be visualized as follows, with $\mathbb{C}_N$ in the top row and $\mathbb{C}_S$ in the bottom. This diagram is a good starting point to construct explicit examples:

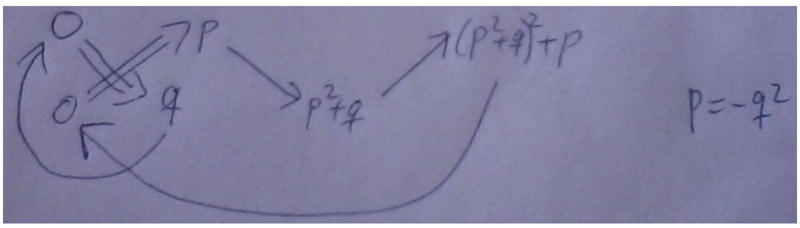


(9)

Example 2.2 (Disjoint cycles of periods 4 and 2)
To determine quadratic rational maps f, such that the critical point 0 is 4-periodic and ∞ is 2-periodic, consider $f(z) = f_b(z) = \frac{z^2+b}{z^2-1}$ according to Section 6: then $\infty \Rightarrow 1 \to \infty$ and $0 \Rightarrow -b \to \frac{b}{b-1} \to \dots$. Setting $f_b^4(0) = 0$ gives a polynomial equation for the parameter, $b(b+1)(b^4 - 3b^3 + 6b^2 - 4b + 1) = 0$. Here $b = 0$ corresponds to period 1 and $b = -1$ does not define a quadratic rational map, but the four complex solutions are meaningful.

To determine matings with the same branch portrait, we need $P \sqcup Q$ such that P is 4-periodic and Q is 2-periodic, so $Q(z) = z^2 - 1$. The equation $P^4(0) = 0$ gives four real and four complex parameters. All of the real solutions have lower periods or the mating is obstructed, so only four matings remain; these give four distinct rational maps, and all maps from the previous paragraph are realized as geometric matings. E.g., the Kokopelli polynomial with $p \approx -0.156520 + 1.032247\,\mathrm{i}$ corresponds to $b \approx 1.052038 + 1.657938\,\mathrm{i}$. This mating is visualized in Figure 1.

To find anti-matings with periods 4 and 2, start with (9). Now the relation $P(Q(0)) = 0$ gives $q^2 + p = 0$. Setting $p = -q^2$ in (10), the condition $Q(P(Q(P(0)))) = 0$ becomes $(q^8 + 2q^5)^2 + q = 0$, or with $x = q^3 \neq 0$ we have $(x+1)(x^4 + 3x^3 + x^2 - x + 1) = 0$. Here $x = -1$ has period 2.



(10)

The other two real solutions are obstructed, see Example 3.3. Two complex solutions remain; for $q^3 \approx 0.339 - 0.447\,\mathrm{i}$ and $p = -q^2$, the geometric anti-mating gives the *same rational map as the mating of Kokopelli and Basilica, see Figure 2. So the matings of satellite period-4 with the Basilica cannot be realized as anti-matings.*

In Example 2.2, an equation for q^3 was obtained. This corresponds to the fact that q may be replaced with $\zeta^{-1}q$, $\zeta^3 = 1$, without changing the dynamics, since $p = -q^2$ is replaced with ζp at the same time. — Note that matings are parametrized by pairs of independent polynomials, while anti-matings reflect the dynamics of even quartic polynomials $Q \circ P$. To realize a given branch portrait, p and q are determined from two coupled equations. Further examples are discussed in the following Sections 2.4 and 2.5.

2.4 Defining rational maps

Each external ray of a formal mating $g = P \sqcup Q$ or anti-mating $g = P \sqcap Q$ connects two landing points, one in each Julia set. When a point has two or more rays landing, a graph is obtained, with rays alternating between endpoints in one Julia set and in the other one. A **ray-equivalence class** is a maximal connected set formed by rays and landing points, or a single point in a Fatou component. A periodic ray-equivalence class has at most one branch point in the case of mating [34, 17], because the branches of a periodic point are permuted transitively by the first-return map; this may be different for anti-matings [1]. The combinatorics of ray connections is discussed further in [17, 19, 18].

Now suppose that the ray-equivalence relation $\sim$ is closed: the limits of equivalent sequences are contained in a single ray-equivalence class. Then the quotient $\widehat{\mathbb{C}}/\sim$ is a Hausdorff space. If every ray-equivalence class is a tree, this quotient is a topological sphere according to the Moore Theorem [29]. Then the **topological mating** $P \coprod Q$ or **topological anti-mating** $P \prod Q$ is a branched cover of degree 2 on the sphere. Note that it is defined only up to conjugation, and that this sphere has a natural complex structure only on Fatou components. This definition is motivated by the observation of Douady–Hubbard [10], that certain rational maps have a Julia set structured as two polynomial Julia sets glued together at complex conjugate angles. An equivalent definition would be to consider the ray-equivalence relation on the disjoint union $\mathcal{K}_p \uplus \mathcal{K}_q$ or $\mathcal{K}_{QP} \uplus \mathcal{K}_{PQ}$; this gives the same quotient space, but starting with the formal (anti-)mating has various advantages: we may look at the geometry of ray-equivalence classes and speak of trees and loops, we may employ the Moore Theorem, and we may ask whether a projection to the quotient space is a limit of homeomorphisms or a pseudo-isotopy.

Now a **geometric (anti-)mating** is a rational map f conjugate to the topological (anti-)mating. The conjugation shall be conformal on Fatou components, which determines f up to Möbius conjugation; this uniqueness statement assumes some rigidity properties of quadratic rational maps, which have been verified in the cases under consideration here, and which are conjectured to hold in general. Then f is determined uniquely by the following normalization: the points 0, ∞, and 1 of the formal (anti-)mating are mapped to the same values again; so 0 corresponds to the critical point of P, ∞ to the critical point of Q, and 1 to the fixed 0-ray. A conjugation respecting this normalization is denoted by $f \cong P \coprod Q$ or $f \cong P \prod Q$, and f is "the" geometric (anti-)mating. We shall write $\widetilde{f} \simeq P \coprod Q$ or $\widetilde{f} \simeq P \prod Q$ when only the critical points are normalized to 0 and ∞ in this way, and 1 is arbitrary;

then $\widetilde{f}$ is conjugate to f by a linear map, a rescaling.

Remark 2.3 (Shared matings and anti-matings)
If $P_1 \coprod Q_1 \simeq f \simeq P_2 \coprod Q_2$ with $P_1 \neq P_2$ or $Q_1 \neq Q_2$, the rational map f has two different representations as a mating. Note that in general $P \coprod Q$ and $Q \coprod P$ are conjugate by a map interchanging the critical points, but not by one respecting the critical points. If $P \coprod Q \simeq Q \coprod P$ with $P \neq Q$, this is considered as a shared mating as well; then the geometric mating is a symmetric rational map. At least in the hyperbolic case, this is called a Wittner flip. Non-hyperbolic shared matings are constructed in [18].

The same notion applies to anti-matings $P_1 \prod Q_1 \simeq f \simeq P_2 \prod Q_2$; since the parameters always come in triples, a shared anti-mating means that for all ζ with $\zeta^3 = 1$, we have $p_1 \neq \zeta p_2$ or $q_1 \neq \zeta^{-1} q_2$.

Here is an equivalent definition of the geometric mating. $h_S \uplus h_N : \mathcal{K}_p \uplus \mathcal{K}_q \to \widehat{\mathbb{C}}$ is continuous and surjective, conformal in the interior, its fibers are ray-equivalence classes, and it is a semiconjugation: $h_S \circ P = f \circ h_S$ and $h_N \circ Q = f \circ h_N$. The geometric anti-mating is described by a semi-conjugation $h_S \uplus h_N : \mathcal{K}_{QP} \uplus \mathcal{K}_{PQ} \to \widehat{\mathbb{C}}$ with $h_N \circ P = f \circ h_S$ and $h_S \circ Q = f \circ h_N$.

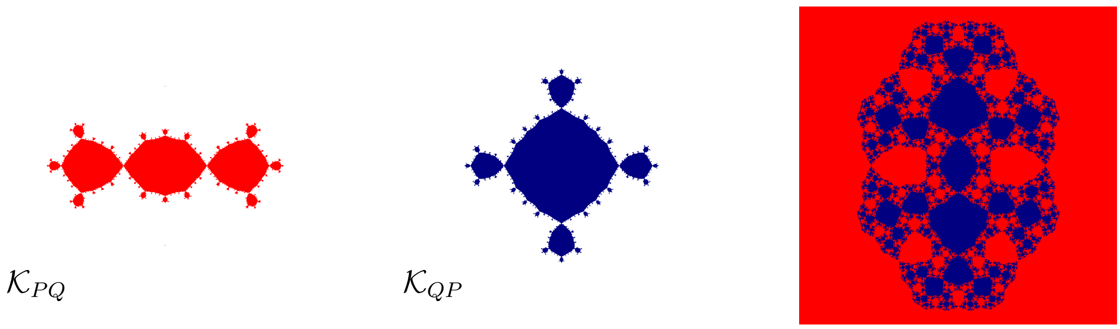


Figure 3: In the anti-mating $f(z) \simeq (z^2) \prod (z^2 - 1)$, the filled Julia sets of $Q \circ P(z) = z^4 - 1$ (middle) and $P \circ Q(z) = (z^2 - 1)^2$ (left) are glued together. The right image shows the geometric anti-mating $f(z) = 1 + \frac{a}{z^2}$ with $a = -(3 + \sqrt{5})/2$, which is bitransitive of period 4. Here the moduli space map is a Basilica polynomial in fact. Note that both Julia sets are real-symmetric, but there are no cyclic ray connections; α_{QP} has the external angles $\pm 6/15$ and α_{PQ} has $\pm 3/15$. This example belongs to the family discussed in Section 5.

With a few exceptions [43, 12, 29, 17], the actual construction of a geometric mating or anti-mating is based on the following approach in the postcritically finite case: the formal mating $g = P \sqcup Q$ or anti-mating $g = P \sqcap Q$ is a Thurston map, a branched cover with a finite marked set. Two Thurston maps are combinatorially equivalent, if they become topologically conjugate after an isotopic deformation relative to the marked sets. Under appropriate conditions, and after collapsing a finite number of ray-equivalence classes, the **essential (anti-)mating** $\widetilde{g}$ is combinatorially equivalent to a rational map, the **combinatorial (anti-)mating** f. Finally, there is a suitable semi-conjugation from g to f, which shows that the topological (anti-)mating exists and f is a geometric (anti-)mating. This approach is carried through, and explained more thoroughly, in Section 3.

There is yet another definition in terms of laminations [31, 1]. A lamination is a collection of hyperbolic geodesics in the unit disk $\mathbb{D}$, which is in a sense forward invariant or completely invariant under angle doubling, such that each leaf represents two rays landing together at a Julia set [39]. Polygons of leaves correspond to branch points and infinite gaps to Fatou components. Now we have laminations in two disk, which are either mapped to themselves or to each other. Identify their boundaries at conjugate angles, then collapsing only postcritical leaves is equivalent to the essential (anti-)mating, while collapsing all leaves of closed invariant laminations gives the topological (anti-)mating.

Remark 2.4 (Higher degrees)
The various definitions of (anti-)mating have a direct generalization to polynomials and rational maps of degree $d > 2$. There are more parameters, more critical points, more fixed points, and corresponding changes regarding the normalization and possible rotation. And there are new phenomena:

• *When the degree d is a square, the rational map may be a flexible Lattès map [27]. Then (anti-)mating is not unique, in the sense that a one-parameter family of geometric (anti-)matings corresponds to a single essential mating. For quadratic maps, this does not happen in the postcritically finite case, and probably not at all according to the Dense Hyperbolicity Conjecture.*

• *The existence results in Section 3 rely on the description of Thurston obstructions in terms of Lévy cycles in the quadratic case. This is no longer true for $d > 2$ [36, 6].*

2.5 Rational maps as matings and anti-matings

Hyperbolic quadratic rational maps f are classified as follows according to Rees [31] and Milnor [25]:

B or II is **bitransitive**: both critical points are in the same cycle of Fatou components but not in the same component.

C or III is a **capture**: one critical point is in a strictly preperiodic Fatou component. (Rees uses "capture" for a subset of type III-maps, with a restricted capture path.)

D or IV has two **disjoint** cycles of Fatou components.

E or I is **escaping**: both critical orbits converge to a fixed point within the only Fatou component.

Now each hyperbolic component of type B, C, D contains a unique postcritically finite map up to normalization, but there is no such map of type E. For any postcritically finite map f, the branch portrait is a directed graph, whose vertices correspond to critical and postcritical points. If f is not hyperbolic, either one or both critical points are iterated to repelling periodic points; in the latter case, the critical orbits may be disjoint, or meet in various ways.

Proposition 2.5 (Types of rational maps)
Consider a hyperbolic quadratic rational map f.

1. *If $f \cong P \coprod Q$ is a geometric mating, then f is of type D. All combinations of periods ≥ 1 occur, except for both critical orbits of period 2.*

2. *If $f \cong P \prod Q$ is a geometric anti-mating, then f may be of type B, C, or D. All attracting periods are even, and in types B and C, the number of iterations from one critical point to the other one is odd.*

Proof: 1. The interiors of the two filled Julia sets are disjoint. We have period 1 for parameters in the main cardioid of $\mathcal{M}$, all periods ≥ 2 in the 1/2-limb, and all periods ≥ 3 in the 1/3-limb.

2. The parity is a consequence of the orbits alternating between two filled Julia sets. See Figure 2 for type D, Figure 3 for type B, and Section 6 for type C. ■

Note that in non-hyperbolic cases, due to possible identifications, the parity is not restricted: a ray-equivalence class may have a lower period than the landing points of the rays, and points in different Julia sets are identified. — Not every f with the properties of Proposition 2.5 is a geometric mating or anti-mating, respectively. See Section 4 for a sufficient criterion. For anti-matings, counterexamples are found in Section 7 for type B, Section 6 for type C, and in Example 2.2 for type D. For matings, the classical counterexample is due to Ben Wittner [42, p. 84]:

Example 2.6 (Wittner)
There is a unique real quadratic rational map of the form $f_w(z) = (z^2+a)/(z^2+b)$*, such that* 0 *is 4-periodic and* ∞ *is 3-periodic; approximately* $a = -1.381186$ *and* $b = -0.388205$*. This map is not a geometric mating of quadratic polynomials.*

Proof: Any mating $f \simeq P \coprod Q$ has this branch portrait, if and only if P is 4-periodic and Q is 3-periodic. Wittner determined all combinations numerically and found them to be different from f_w. Alternatively, a combinatorial argument shows that for all of these matings, some periodic Fatou components have common boundary points: this is obvious when P or Q is of satellite type. Otherwise P is the (Co-)Kokopelli with angle $\pm 3/15$ and Q is the Airplane — and 4-periodic Fatou components are drawn together pairwise by ray-connections through the 2-cycle of the Airplane. But for f_w no closed Fatou components in the same cycle meet, since the critical orbits are ordered cyclically as $z_3 < w_2 < z_0 < z_2 < w_1 < z_1 < w_0$ on $\mathbb{R} \cup \{\infty\}$. The Julia set $\mathcal{J}_w$ is a Sierpinski carpet in fact [25]. ■

Periodic points of the topological and geometric (anti-)matings correspond to periodic ray-equivalence classes of the formal (anti-)mating. For periods 1 and 2, these have the following form:

Proposition 2.7 (Ray-equivalence classes)
1. *Suppose there are no direct connections between the fixed points* $\varphi_0(\alpha_p)$ *and* $\varphi_\infty(\alpha_q)$ *of the formal mating* $g = P \sqcup Q$*, and consider* $f \cong P \coprod Q$*. Then the fixed points of* f *correspond to the* 0*-ray and to the ray-equivalence classes of both* α*'s, which may have various rotation numbers and connections of diameters* 2 *or* 4*. Or, one or both may be attracting.*

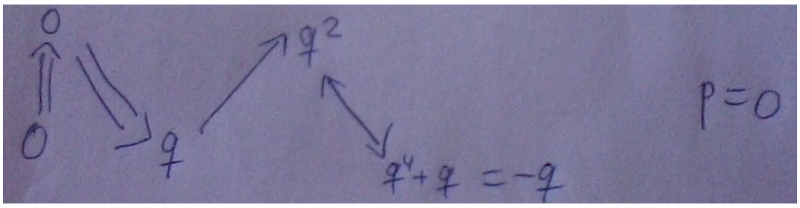
The 2*-periodic classes are given by the* $1/3$*- and* $2/3$*-rays, except when, say,* p *is in the* $1/2$*-limb: then the ray-equivalence classes contain the* 2*-periodic points of* P*, or these are attracting, while the* 2*-cycle of* Q *is identified with* α_p*.*

2. *Suppose there are no direct connections between* $\varphi_0(\alpha_{QP})$ *and* $\varphi_\infty(\alpha_{PQ})$ *in the formal anti-mating* $g = P \sqcap Q$*, and* $f \cong P \prod Q$*. Then the fixed points of* f *correspond to the* 0*-ray,* $1/3$*-ray, and* $2/3$*-ray. The* 2*-cycle is given by the ray equivalence classes of both* α*'s, which may have various rotation numbers and connections of different diameters, or be attracting.*

Proof: The closed ray-equivalence classes are mutually disjoint and they have the required periods. The 0-ray of the mating, and the three fixed rays of the anti-mating, are the only fixed rays; so they cannot land together with rays of a higher ray period. Finally, a quadratic rational map has only three fixed points and one 2-cycle, with possible identifications for parabolic maps, so there are no more ray-equivalence classes with these periods. For the classes of α_p and α_q, recall that the root of a limb of $\mathcal{M}$ is narrow: all other components in the wake have a higher period. ∎

Note that under these combinatorial assumptions, in the geometrically finite or postcritically finite case, the geometric (anti-)mating f exists according to Theorems 3.1 and 3.2, but we shall take that as an additional hypothesis for the time being. Proposition 2.7 is employed in Section 3.5 and in the following example:

Example 2.8 (Lattès map of type $(2, 4, 4)$**)**
The quadratic Lattès map f with orbifold type $(2, 4, 4)$ has a well-known representation as a mating $\pm 1/4 \coprod 1/2$; further representations are discussed in Section 5 of [19]. It is given by $f(z) = -1 + \frac{2}{z^2}$ with $0 \Rightarrow \infty \Rightarrow -1 \to 1 \uparrow$, which is conjugate to $f_{-2}(z) = 1 - \frac{2}{z^2}$ in the normalization from Section 5. To represent f as a geometric anti-mating, we use that the fixed points of the rational map correspond to ray-equivalence classes with just one direct connection, between 2-periodic points of the formal anti-mating. So there is no indirect ray connection between $p \in \mathcal{K}_{PQ}$ and $0 \in \mathcal{K}_{PQ}$ either, but $p = 0$:



(11)

These relations have the solution $p = 0$ and $q^3 = -2$, which is unique except for the usual rotation with ζ, $\zeta^3 = 1$. A 2-cycle of g is collapsed indeed in the combinatorial mating; convergence of rational maps is obtained explicitly from the linear order of marked points and the pullback relation of the Thurston Algorithm. Here the moduli space map is a Chebyshev polynomial in fact. Note that both Julia sets $\mathcal{K}_{QP}$ and $\mathcal{K}_{PQ}$ are real-symmetric (or rotated), but there are no cyclic ray connections; see Figure 3 for another example of this phenomenon.

2.6 Implementation of slow mating and anti-mating

formulas, extra care with branch portrait, mention equi gluing

3 Constructing rational maps

Necessary and sufficient conditions for the existence of geometrically finite matings, and of postcritically finite anti-matings, are formulated in Section 3.1. They are proved in Sections 3.2–3.6 using the notion of a good Lévy cycle [38], with special treatment of Lattès maps of type $(2, 2, 2, 2)$. A few remarks on the postcritically infinite case are given in Section 3.7.

3.1 Existence of matings and anti-matings

For matings, the classical result is due to Rees–Shishikura–Tan [31, 38, 35], with contributions by Cui, Haïssinsky, and the author [13, 9, 19], based on work of Douady–Hubbard and Thurston [11, 14]:

Theorem 3.1 (Quadratic mating)
Suppose $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ are geometrically finite, i.e., postcritically finite, hyperbolic, or parabolic. Then the geometric mating $f \cong P \coprod Q$ exists, if and only if the parameters p and q are not from conjugate closed limbs of $\mathcal{M}$.

A corresponding result for postcritically finite anti-matings is due to Ahmadi Dastjerdi, as a special case of Theorem 5.11.1 in [1], using an alternative definition in terms of laminations:

Theorem 3.2 (Quadratic anti-mating)
Suppose $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ are such that the quartic polynomial $Q \circ P$ is postcritically finite. Then the geometric anti-mating $f \cong P \prod Q$ exists, if and only if the rays with angles 0, $1/3$, and $2/3$ land at distinct fixed points of $Q \circ P$.

There is still a gap, or possible exception, for $\widetilde{g}$ of Lattès type $(2, 2, 2, 2)$ with a postcritical 2-cycle.

In the postcritically finite case, both theorems will be proved in parallel in Sections 3.2–3.6, by showing that the essential (anti-)mating $\widetilde{g}$ is an unobstructed Thurston map and the combinatorial (anti-)mating $f \sim \widetilde{g}$ exists, which requires a special treatment of Lattès-type maps. Morover, there is a semi-conjugation from g to f collapsing all ray-equivalence classes, so f is a geometric (anti-)mating in fact and the topological (anti-)mating exists as well, $f \cong P \coprod Q$ or $f \cong P \prod Q$.

Example 3.3 (An anti-mating with a Hausdorff obstruction)
According to Example 2.2, there are two formal anti-matings $g = P \sqcap Q$ with real parameters, such that ∞ is 2-periodic and 0 is 4-periodic: we have $q^3 \approx -1.3894$ or $q^3 \approx -2.2888$, and $p = -q^2$. In both cases, $(Q \circ P)'(\alpha_{QP}) > 1$, so both $\mathcal{R}_{QP}(\pm 1/3)$ land at α_{QP}, and there is a good obstruction and a closed ray connection between $\varphi_0(\alpha_{QP})$ and $\varphi_\infty(\alpha_{PQ})$. Gluing the quartic filled Julia sets gives a countable union of spheres for $q^3 \approx -1.3894$. When $q^3 \approx -2.2888$, the closed ray connection with the angles $\pm 11/24$ separates the critical values. Suitable preimages accumulate on the boundary of the Fatou component around $\varphi_\infty(p)$, so the ray equivalence relation is not closed, and the quotient space is not Hausdorff. — For matings, examples of this kind are discussed in terms of renormalization in [17].

3.2 Obstructions and Lévy cycles

Thurston Theorem, later developments: additional, without orbifold, algebraic, augmented

always marked crpt and 1

obstructions, Lévy, good and degenerate, check vs removable

ref tree in talk Mitsu

single good orientation-reversing on TL p. 15, equivalent dfn on p. 14

3.3 Pullback converges to ray connections

minimal number of rays crossing equator converge to rays, other connected components to points, also with additional marked points, extremities marked because minimal number, homotopy wrt P' defines unique preimage

neighborhoods $V \supset V' \supset \mathcal{K}_{PQ} \cup \mathcal{K}_{QP}$ with $g : V' \to V$ expanding

pull back a connected set

also for additional marked points, also deg Levy

3.4 Rotation numbers for good obstructions

For matings, condition of conjugate rotation numbers, which means conjugate limbs.

For anti-maings, alternative proof with laminations: Theorem 5.11.1 in [1]

A non-removable obstruction of the formal anti-mating would be related to cyclic ray connections between fixed points of $Q \circ P$ and $P \circ Q$. It turns out that these are of a quite restricted form:

Proposition 3.4 (Cyclic ray-connections)
There is a cyclic ray connection between fixed points of $Q \circ P$ and $P \circ Q$, if and only if two of the rays with angles 0, $1/3$, and $2/3$ land together at $\partial\mathcal{K}_{QP}$.

Proof: First, suppose that the rays with angles, say, 0 and 1/3 land together at a fixed point $\alpha_{QP} \in \partial\mathcal{K}_{QP}$. Then the rays with angles $2 \cdot 0 = 0$ and $2 \cdot 1/3 = 2/3$ land together at a fixed point $\alpha_{PQ} \in \partial\mathcal{K}_{PQ}$ and the angles are conjugate, so there is a closed ray connection. If the fixed points are not marked, this loop is a good Lévy cycle of period 1, since it is mapped to itself in an orientation-reversing way by the formal anti-mating $g = P \sqcap Q$. If the fixed points are marked, two nearby loops form a good Lévy cycle of period 2.

Conversely, assume that the three rays land at distinct fixed points, then the fourth fixed point α_{QP} is attracting or a pinching point with rotation number $\neq 0$. Now $P : \alpha_{QP} \mapsto \alpha_{PQ}$ and $Q : \alpha_{PQ} \mapsto \alpha_{QP}$ doubles the angles. We shall try to find a cyclic ray connection, and fail. The graph of ray connections will be turned by g^2, so the rotation numbers at α_{QP} and α_{PQ} shall be conjugate, and they are equal by the semi-conjugations. Thus the rotation number is $1/2$ and all angles have denominator 15; these form six possible 2-cycles.

If, e.g., there is one 2-cycle of rays at α_{QP} with the angles 1/15 and 4/15, then α_{PQ} has 2/15 and 8/15. These are not conjugate to the former ones, and this is not a closed ray connection. So suppose that there are two 2-cycles at α_{QP}, one with the angles 1/15 and 4/15, another one with 7/15 and 13/15. Then the angles of α_{PQ} would be conjugate, but unfortunately, these four angles cannot be landing at the same point of a symmetric Julia set.

The same arguments work for all six possible pairs of angles, so there cannot be a direct cyclic ray connection. Again, suppose that α_{QP} has the angles 1/15 and 4/15, and α_{PQ} has 2/15 and 8/15. To rule out indirect ray connections as well, recall that $P \circ Q$ is even. So $-\alpha_{PQ}$ has the angles 19/30 and 1/30. The complex conjugate angles 11/30 and 29/30 probably belong to different points in $\mathcal{K}_{QP}$, but the rays are connected through $\mathcal{K}_{PQ}$ and they shield the angles 1/15 and 4/15 from forming indirect ray connections: these would be invariant under the rotation as well and the denominator would have to be 15 throughout. Actually, indirect ray connections are ruled out, because the intersection with each Julia set is connected, thus a single point. ∎

3.5 Combinatorial mating and anti-mating

when no good obstruction, collapse ray-equivalence trees to define the essential (anti-) mating. alternative definitions. note Julia sets no longer invariant.

show it is unobstructed.

then there is a combinatorial equivalence to a rational map

however, this requires special treatment of $(2, 2, 2, 2)$. See [19] for matings.

Here for anti-matings: show that a)b)c) only self-anti-matings, d) still open.

convergence, automatic identification, also additional

3.6 Geometric mating and anti-mating

[35] with rays, [8] with continua

3.7 Postcritically infinite maps

also Siegel, Timorin boundary, Dima [12, 29, 17]

4 Equator and anti-equator

The characterization of matings by an equator is a folk theorem going back to Thurston; it was proved in [42, 24] under similar assumptions as below. The corresponding result for anti-matings was conjectured by Meyer [4, 24]. Here there is nothing special about the quadratic case, and we shall treat the case of degree $d \geq 2$ according to Remark 2.4:

Theorem 4.1 (Equator and anti-equator)
Suppose f is a postcritically finite rational map of degree $d \geq 2$, with marked critical points, and possibly additional marked points. An ***equator*** *is a simple closed curve γ, such that $\gamma' = f^{-1}(\gamma)$ is connected and isotopic to γ relative to the marked set, and $f : \gamma' \to \gamma$ is orientation-preserving with respect to the isotopy. When f is orientation-reversing, γ is an* ***anti-equator****.*

1. *Now f is combinatorially equivalent to a formal mating $g = P \sqcup Q$, if and only if it has an equator.*

2. *And f is combinatorially equivalent to a formal anti-mating $g = P \sqcap Q$, if and only if it has an anti-equator.*

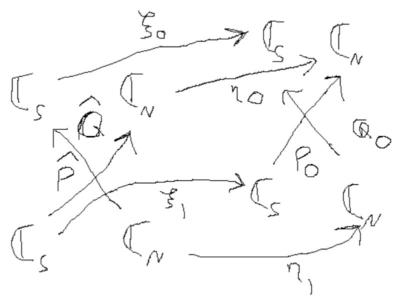
Remark 4.2 (Hyperbolic and non-hyperbolic maps)
1. Suppose $f \cong P \coprod Q$ is a postcritically finite geometric mating, without additional marked points. If f is hyperbolic, it is combinatorially equivalent to the formal mating $g = P \sqcup Q$, so it has an equator. If f is not hyperbolic, there may be identifications from postcritical ray-equivalence classes, such that g is obstructed and f is combinatorially equivalent to an essential mating $\widetilde{g} \neq g$. Then f does not have an equator corresponding to this representation as a mating. — Analogous statements apply to anti-matings.

2. In the hyperbolic case, shared matings may be obtained by finding non-homotopic equators for the same f. When f is not hyperbolic, there may be more representations as a mating, than there are equators. See [18, 24] for various techniques to obtain shared matings in this case.

Proof of Theorem 4.1: By construction, a formal mating $g = P \sqcup Q$ has the equator $\partial\mathbb{D}$, and a formal anti-mating $g = P \sqcap Q$ has the anti-equator $\partial\mathbb{D}$. So if f is combinatorially equivalent to g, with $\psi_0 \circ g = f \circ \psi_1$, then $\gamma = \psi_0(\partial\mathbb{D})$ is isotopic to $\gamma' = f^{-1}(\gamma) = \psi_1(\partial\mathbb{D})$.

1. Conversely, assuming that f has an equator, it is combinatorially equivalent to a Thurston map g with $g(z) = z^d$ for $z \in \partial\mathbb{D}$. So g is a formal mating of two topological polynomials $\widehat{P} = \varphi_0^{-1} \circ g \circ \varphi_0$ and $\widehat{Q} = \varphi_\infty^{-1} \circ g \circ \varphi_\infty$. Suppose $\widehat{P}$ is obstructed, thus f is obstructed as well, then f would be a flexible Lattès map with four postcritical points. Now $\widehat{P}$ and $\widehat{Q}$ together have six postcritical points including ∞; since $\widehat{P}$ has at least four, $\widehat{Q}$ has at most two. Then $d-1$ finite critical points of $\widehat{Q}$ are mapped to the finite postcritical point, which is fixed and has $2d-1 > d$ preimages, a contradiction. So $\widehat{P}$ and analogously $\widehat{Q}$ are unobstructed in any case. By the Thurston Theorem, there are equivalent polynomials P and Q, which are determined uniquely by requiring them monic, centered, and with suitable asymptotics of the 0-ray under the equivalence. Now g is equivalent to the formal mating $P \sqcup Q$: if the equivalences did not match on the equator, the 0-ray would be Dehn twisted under pullback.

2. Likewise, when f has an anti-equator, it is equivalent to a Thurston map g with $g(z) = z^{-d}$ for $z \in \partial\mathbb{D}$. So g is a formal anti-mating of two topological polynomials $\widehat{P} = \varphi_\infty^{-1} \circ g \circ \varphi_0$ and $\widehat{Q} = \varphi_0^{-1} \circ g \circ \varphi_\infty$. By item 1, $\widehat{Q} \circ \widehat{P}$ and $\widehat{P} \circ \widehat{Q}$ are unobstructed and equivalent to polynomials of degree d^2, but we want to know more. Analogously to Section 2.3, define the disjoint union $\widehat{\mathbb{C}}_S \uplus \widehat{\mathbb{C}}_N$ and consider $\widehat{P} : \widehat{\mathbb{C}}_S \to \widehat{\mathbb{C}}_N$ and $\widehat{Q} : \widehat{\mathbb{C}}_N \to \widehat{\mathbb{C}}_S$. For a pair of homeomorphisms $\xi_0 : \widehat{\mathbb{C}}_S \to \widehat{\mathbb{C}}_S$ and $\eta_0 : \widehat{\mathbb{C}}_N \to \widehat{\mathbb{C}}_N$, a kind of Thurston pullback is defined naturally by the commuting diagrams



$$; \qquad (12)$$

it descends to a pullback map on a pair of Teichmüller spaces. For a choice of $(\xi_0\,,\,\eta_0)$, the pullback defines a sequence of $(\xi_j\,,\,\eta_j)$. Now both subsequences ξ_{2k} and ξ_{2k+1} correspond to the pullback map for $\widehat{Q} \circ \widehat{P}$, and their classes in Teichmüller space converge to the same element. In the second Teichmüller space, the classes of η_j converge for the same reason. So the marked points and hence the polynomials P_j and Q_j converge as well: for appropriate $(\xi\,,\,\eta)$ we have $\xi \circ \widehat{Q} = Q \circ \eta'$ and $\eta \circ \widehat{P} = P \circ \xi'$, with ξ' isotopic to ξ relative to the marked points in $\widehat{\mathbb{C}}_S$ and η' isotopic to η relative to the marked points in $\widehat{\mathbb{C}}_N$.

For the pullback it is easiest to send chosen marked points to 0, 1, ∞. After obtaining the limits, we may assume that ξ and η have affine asymptotics at ∞. Perform affine conjugations according to (7), such that P and Q are monic and centered, and such that ξ and η do not rotate the 0-rays. Then the equivalences match on the equator without an additional Dehn twist, and g is combinatorially equivalent to the formal anti-mating $P \sqcap Q$. — The choices made in the construction of g from f implicitly singled out one of the $d+1$ fixed points to be on the 0-ray; afterwards, g may be rotated with ζ, $\zeta^{d+1} = 1$. ∎

When P and Q have only preperiodic critical points, the essential mating $\widetilde{g}$ and the geometric mating $f \cong P \amalg Q$ may have a **pseudo-equator**, which passes through all postcritical points; see [23, 24] for the definition. The equator of g is deformed to a pseudo-equator of $\widetilde{g}$, if and only if there are at most direct ray connections between postcritical points. Conversely, when f has a pseudo-equator γ, each pseudo-isotopy from γ to $f^{-1}(\gamma)$ determines a pair of polynomials P, Q with $f \simeq P \amalg Q$. — Probably there are analogous statements for anti-matings and anti-pseudo-equators.

5 Bitransitive maps

expand: Suppose $p = 0$, so $Q \circ P(z) = z^4 + q$ and $P \circ Q(z) = (z^2 + q)^2$. The connectedness locus in the q-plane is the Multibrot set of degree 4. If the geometric anti-mating f of $P(z) = z^2$ and $Q(z) = z^2 + q$ exists, it may be normalized to the form $f(z) = f_a(z) = 1 + \frac{a}{z^2}$. These quadratic rational maps satisfy $\dots \to 0 \Rightarrow \infty \Rightarrow 1 \to \dots$; the parameter space is shown in Figure 4 right. When f_a is hyperbolic, it must be of bitransitive type (or escaping), so it cannot be a mating. (Preperiodic matings in this family provide a simple case of mating discontinuity. See Theorem 6.1 in [18].) Conversely, if f_a is an anti-mating, then $p = 0$ in the hyperbolic case; in the preperiodic case we may have $p \neq 0$, when there is a ray connection between $p \in \mathcal{K}_{PQ}$ and $0 \in \mathcal{K}_{PQ}$. Concrete examples have been given in Example 2.8 and in Figure 3.

6 Basilica maps in V_2

expand: $p = -q^2$ gives $Q \circ P(z) = (z^2 - q^2)^2 + q$ and $P \circ Q(z) = z^4 + 2qz^2$. This quartic family is discussed by Gamaliel Blé [2]. If the geometric anti-mating f of $P(z) = z^2 - q^2$ and $Q(z) = z^2 + q$ exists, it is in V_2: $f(z) = f_b(z) = \frac{z^2+b}{z^2-1}$ has a 2-periodic critical point $\infty \Rightarrow 1 \to \infty$. This example of anti-matings is due to Timorin [40] in a different normalization. All hyperbolic maps of disjoint type are known to be matings $S \amalg B$ with the Basilica polynomial $B(z) = z^2 - 1$ [...]. Some maps of disjoint type or capture type are anti-matings as well. See the example of disjoint type in Figure 2. Examples of capture type require at least 5 steps from 0 to ∞; probably the blue components in Figure 5 right contain anti-matings, when they are connected to the outer bitransitive component through a series of small Mandelbrot sets. Finally, an anti-mating of Misiurewicz type is given by $q^3 = \sqrt{2} - 1$ and $b = 2$; the same map is a mating $(z^2 \pm \mathrm{i}) \amalg (z^2 - 1)$.

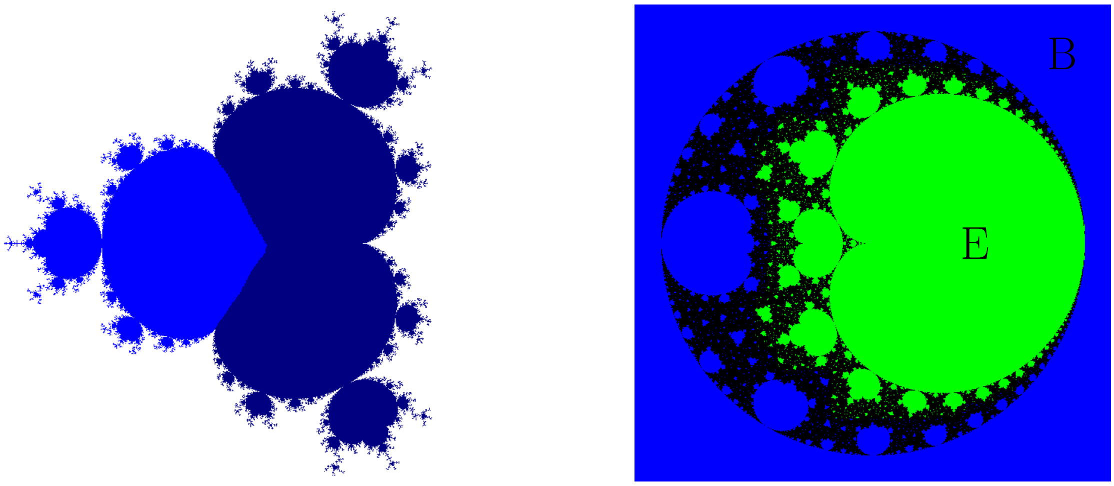


Figure 4: Left: quartic polynomials $P \circ Q$ or $Q \circ P$ with $P(z) = z^2$ and $Q(z) = z^2 + q$, the connectedness locus is the Multibrot set of $z^4 + q$. Right: rational maps f_a with $0 \Rightarrow \infty$. Probably the exterior disk with its blue and black sublimbs corresponds to one-third of the Multibrot set, or to that set parametrized by q^3.

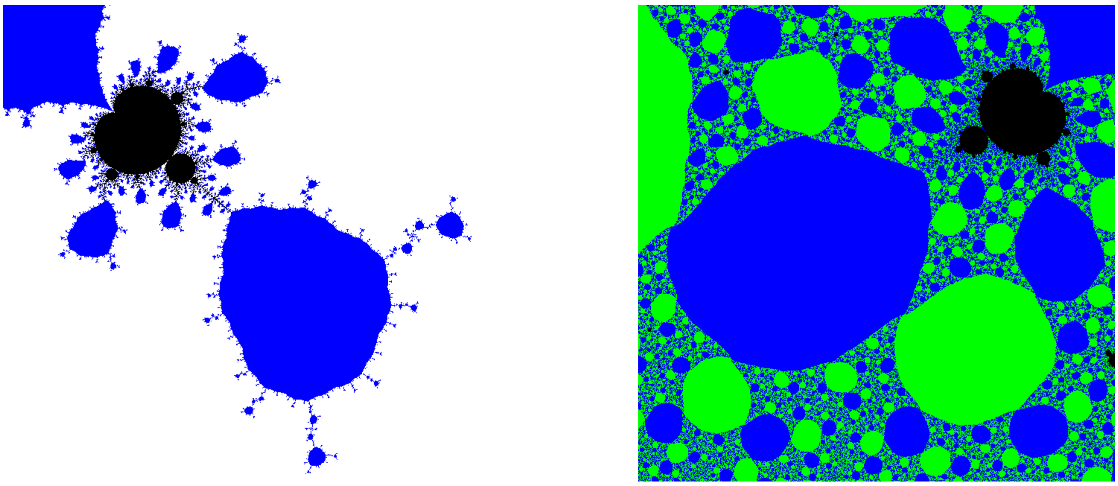


Figure 5: Corresponding subsets of the parameter planes, left: quartic polynomials $P \circ Q$ or $Q \circ P$ with $P(z) = z^2 - q^2$ and $Q(z) = z^2 + q$, right: rational maps f_b with ∞ 2-periodic. Anti-mating maps the polynomial connectedness locus into the rational family. Note that all capture components of the polynomial family are blue, while capture components in V_2 are blue or green; only a subset of the blue ones contains anti-matings.

7 Symmetric maps and Chebyshev matings

A normalized quadratic rational map commutes with the involution $j(z) = 1/z$, if it is of the form $f_c(z) = \frac{z^2+c}{1+cz^2}$ with $c \neq \pm 1$; for $c \to \infty$ it becomes $1/z^2$. Every self-mating $f \cong P \coprod P$ is symmetric; for suitable $P \neq Q$ there are also flipped matings $P \coprod Q \simeq f \simeq Q \coprod P$, which are symmetric as well. Moreover, any self-anti-mating $f \cong P \prod P$ is symmetric; here the quadratic dynamics of P is meaningful, we have two copies of the quadratic Julia set $\mathcal{K}_p$ glued together, and the critical orbits alternate between both copies. If $q = \zeta p$ with $\zeta^3 = 1$, then $f \cong P \prod Q$ is a rescaled self-anti-mating.

Note that $j \circ f_c = f_c \circ j$ implies that $f_{\widetilde{c}} = j \circ f_c$ is a symmetric map as well; we have $\widetilde{c} = 1/c$. The Julia sets and the second iterates of f_c and $f_{\widetilde{c}}$ agree. Now $i(c) = 1/c$ defines an involution in parameter space. Since the critical orbits are interchanged by the dynamic involution j, exchanging every other point in the critcal orbits of f_c gives the critical orbits of $f_{\widetilde{c}}$. So i acts as follows on hyperbolic symmetric maps:

- Disjoint type D with even periods is transformed to the same kind;
- Type D with odd periods n is transformed to bitransitive type B with period $2n$, and vice versa;
- Type B with period divisible by 4 is transformed to the same kind.

Moreover, escaping type E is transformed to itself, and there are no symmetric maps of capture type C. Another transformation due to Carsten Petersen is discussed below. Here the normalizations are adapted to (anti-)matings with critical points at 0 and ∞, but transformations of symmetric maps have simpler formulas in a different normalization. For comparison, these formulas are presented here using capital letters; they are related by $Z = \frac{1+z}{1-z}$ and $z = \frac{Z-1}{Z+1}$, $C = \frac{1}{2}\,\frac{1+c}{1-c}$ and $c = \frac{2C-1}{2C+1}$.

dynamic involution	$j(z) = 1/z$	$J(Z) = -Z$
symmetric map	$f_c(z) = \frac{z^2+c}{1+cz^2}$	$F_C(Z) = C(Z + 1/Z)$
parameter involution	$i(c) = 1/c$	$I(C) = -C$
dynamic Petersen	$w = l(z) = \frac{2z}{z^2+1}$	$W = L(Z) = Z^2$
Chebyshev map	$y_a(w) = \frac{-w^2+(a+2)}{w^2+a}$	$Y_A(W) = A(W + 1/W + 2)$
parameter Petersen	$a = h(c) = -2\frac{c^2+1}{(c-1)^2}$	$A = H(C) = C^2$

Here a Chebyshev map $y_a(w) = \frac{-w^2+(a+2)}{w^2+a}$, $a \neq -1$, is a quadratic rational map with $\infty \Rightarrow -1 \to 1 \uparrow$. The Petersen transformation $w = l(z)$ identifies z and $j(z)$; it is a semi-conjugation $l \circ f_c = y_a \circ l$ with $a = h(c)$. If f_c is hyperbolic of type D, y_a has the same period, while the period of type B is halved. For matings we have the following relations [25, 43]:

Theorem 7.1 (Petersen transformation)
Consider quadratic polynomials P with $p \in \mathcal{M}$, $\mathcal{K}_p$ locally connected, and symmetric maps f_c with $c \neq \pm 1$.

1. *We have $f_c \cong P \coprod P$, if and only if $f_{\widetilde{c}} \cong P \prod P$ with $\widetilde{c} = 1/c = i(c)$.*
2. *Then both f_c and $f_{\widetilde{c}}$ are semi-conjugate to $y_a(w)$ with $a = h(c) = h(\widetilde{c})$, and $y_a \cong P \coprod T$ with the Chebyshev polynomial $T(z) = z^2 - 2$.*

Proof: Assuming $\varphi_\infty = j \circ \varphi_0$ according to (1), the formal mating and anti-mating are related by $j \circ (P \sqcup P) = P \sqcap P$.

For both maps, j sends the ray $\mathcal{R}(\theta)$ to $\mathcal{R}(-\theta)$

note l also when not self-mating

note Ch complete classification, how anti?

also for formal and thurston, interpretation circle to segment, length of ray connections

one step is with semi-conjugations, why N=jS, first on formal sphere?

Use an equivalent definition of the geometric mating. $h_S \uplus h_N : \mathcal{K}_p \uplus \mathcal{K}_q \to \widehat{\mathbb{C}}$ is continuous and surjective, conformal in the interior, its fibers are ray-equivalence classes, and it is a semiconjugation: $h_S \circ P = f \circ h_S$ and $h_N \circ Q = f \circ h_N$. The geometric anti-mating is described by a semi-conjugation $h_S \uplus h_N : \mathcal{K}_{QP} \uplus \mathcal{K}_{PQ} \to \widehat{\mathbb{C}}$ with $h_N \circ P = f \circ h_S$ and $h_S \circ Q = f \circ h_N$.

Conversely, if $y_a \cong P \coprod T$ and $a = h(c)$, then either $f_c \cong P \coprod P$ or $f_c \cong P \prod P$. — at least when geometrically finite or assuming existence, can only show topological ■

So if P is geometrically finite, both $f \cong P \coprod P$ and $\widetilde{f} \cong P \prod P$ exist, if and only if $p \in \mathcal{M}$ is not in the closed $1/2$-limb.

Discuss all representations of several examples.

description of parameter space, loci are where?

maybe question weird bifu, generalization of petersen for other Ch-matings

ch-anti ex 244

Three of the four kinds of Lattès maps with orbifold signature $(2, 2, 2, 2)$ are symmetric, and in terms of external angles we have $1/4 \prod 1/4 \simeq 3/4 \coprod 3/4$, $1/6 \prod 1/6 \simeq 11/14 \coprod 11/14$, and $11/14 \prod 11/14 \simeq 1/6 \coprod 1/6$. There are further representations by non-self matings [27, 19].

Note that the symmetry locus contains hyperbolic maps as well, which are neither self-matings nor self-anti-matings. See the green–black locus in Figure 6 left. The corresponding maps in the Chebyshev family may have a description as matings with preperiodic polynomials from suitable limbs, which are partially shared; see Section 4 in [18], and the green–black locus in Figure 6 right.

Denote the Rabbit polynomial by R and the Airplane polynomial by A. Up to complex conjugation, there are two symmetric maps of disjoint type with period 3, which are given by the self-mating $R \coprod R$ and the non-self mating $R \coprod A \sim A \coprod R$. The involution $c \mapsto 1/c$ sends each of these to a symmetric bitransitive map of period 6. The former one is described as a self-anti-mating $R \prod R$, but the latter cannot be described as an anti-mating: when $P \sqcap Q$ maps 0 to ∞ and ∞ to 0 in 3 steps, the only solution with $p \neq q$ turns out to be real, $p^3 = -(3 \pm \sqrt{5})/2$ and $q = 1/p$, $q^3 = -(3 \mp \sqrt{5})/2$. Now $f \cong P \prod Q$ is real with $f(z) \approx \frac{z^2 - 7.0352}{z^2 - 3.2754}$ for "+", while the symmetric bitransitive maps are not real with respect to a complex conjugation.

8 Suggestions for further research

There is a gap, or possible counterexample, in the existence result for anti-matings: when the essential anti-mating $\widetilde{g}$ is of type $(2, 2, 2, 2)$ with a postcritical 2-cycle, does the affine lift have conjugate eigenvalues? **Problem:** find all anti-matings of this type, and check the eigenvalues.

The present paper concentrates on anti-matings $P \prod Q$ with $Q \circ P$ postcritically finite. **Problem:** As the next step, construct anti-matings with hyperbolic and parabolic polynomials, especially for geometric anti-matings of types B and C.

Mating is not jointly continuous with respect to the polynomial parameters. Examples of discontinuity are discussed in [18], both classical ones due to Adam Epstein, and new ones. **Problem:** construct analogous examples of discontinuity for anti-matings.

The existence criterion for postcritically finite (anti-)matings in Section 3 relied on the notion of a good Lévy cycle; for degrees $d > 2$, other kinds of non-removable obstructions are possible. **Problem:** Find conditions for matability and

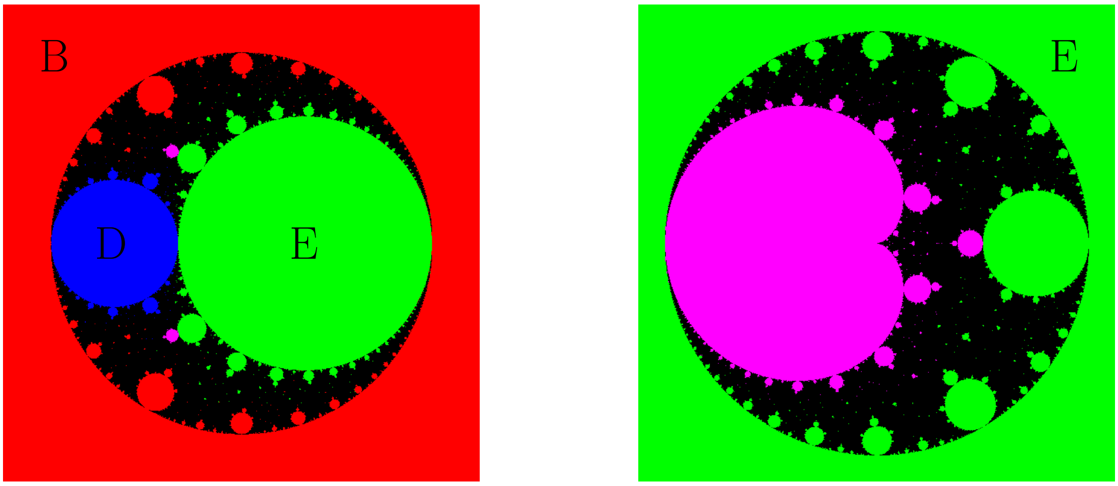


Figure 6: The left image shows the parameter plane of symmetric maps f_c. Conjecturally, the self-mating locus is blue–black and the self-anti-mating locus is red–black; each corresponds to the Mandelbrot set with its 1/2-limb removed.
The Petersen transformation maps both sets to the magenta–black subset of the right image, which is the locus of Chebyshev matings within the family of Chebyshev maps: the rational maps y_a, such that $y_a(\infty)$ is pre-fixed.

anti-matability in higher degrees.

According to Section 2.3, and to Section 5.14 in [1], even quartic polynomials may be modeled by pairs of quadratic polynomials, and pairs of laminations are characterized by pairs of minors. **Problems:**

- Discuss limbs in the two-dimensional parameter space in terms of the rotation numbers and angles at the fixed points α_{QP} and α_{PQ}.
- Describe characteristic ray-pairs [26, 33] of the same ray period within a limb.
- Implement a spider algorithm on two planes, $\mathbb{C}_S \uplus \mathbb{C}_N$, analogously to (12).

For a postcritically finite rational map f with a pseudo-equator, Meyer [24] gives an algorithm to determine polynomials P and Q with $f \simeq P \coprod Q$. **Problem:** find an anlogous algorithm for f with a pseudo-anti-equator.

When a polynomial P has an invariant embedded tree containing the postcritical points $\neq \infty$, with finitely many endpoints or at least compact, its core entropy $h(P)$ is defined as the topological entropy on this Hubbard tree; see [16] and the references therein. It is tempting to define the core entropy of a rational map f in terms of some invariant graph. For $f \cong P \coprod Q$, such a graph is obtained from suitably extended Hubbard trees, and the entropy of f would be $h(f) = \max(h(P), h(Q))$. This approach might be more natural for anti-matings $f \cong P \prod Q$, because the core entropies of $Q \circ P$ and $P \circ Q$ agree: $h(f) = \frac{1}{2}h(Q \circ P) = \frac{1}{2}h(P \circ Q)$. However, in both cases there is an ambiguity due to shared (anti-)matings: for $p, q \in \mathcal{M}$ with the angles 59/240 and 63/240, we have $P \coprod P \simeq Q \coprod Q$ [18], and $P \prod P \simeq Q \prod Q$ according to Theorem 7.1, but $h(P) < h(Q)$. **Problem:** find a useful notion of core entropy for a class of rational maps.

One-parameter families of quartic polynomials and corresponding rational maps are discussed in Sections 5–7. **Problem:** describe the loci of mating and anti-mating within these families. Numerical experiments will be crucial to formulate conjectures.

Symmetric rational maps may be matings of different polynomials $P \neq Q$; the simplest example is $f \simeq R \coprod A \simeq A \coprod R$, with Rabbit R and Airplane A. The Petersen transformation in Theorem 7.1 sends f to $g \simeq A \coprod (z^2 \pm \mathrm{i})$. **Problem:** Is this an example of a more general relation?

The operations of tuning, mating, and anti-mating are special cases of the following kinds of combinations, where one or two polynomials P and Q are inserted into periodic critical Fatou components of a postcritically finite quadratic map R:

1a) R has one periodic critical point and the other one is repelling-preperiodic;

1b) R is of type C;

1c) R is of type D, and $\mathcal{K}_p$ is inserted into one of the two critical components;

2) R is of type D, and $\mathcal{K}_p$ is inserted into one of the two critical components, $\mathcal{K}_q$ into the other one;

3) R is of type B, $\mathcal{K}_{QP}$ and $\mathcal{K}_{PQ}$ are inserted into the critical components.

So tuning of polynomials is of kind 1c), and anti-mating is kind 3 with $R(z) = 1/z^2$. There are two ways to describe mating $P \coprod Q$ here: either set $R = P$, and insert Q into the basin of ∞ according to kind 1a) or 1c); or set $R(z) = z^2$ and insert P and Q according to kind 2).

When the relevant Fatou components are disks, and their closures are pairwise disjoint except for at most one point, these constructions can be analyzed in terms of multicurves [30], and they can be described in terms of renormalization, even when P is not postcritically finite. Closed ray connections and good obstructions must be ruled out in other cases. When the period of R is > 1, Ahmadi Dastjerdi [1] has shown that these do not occur in cases 1b) and 1c), and in case 3) only when there is a closed ray connection through two repelling fixed points; this result includes anti-mating with $R(z) = 1/z^2$ and cases like $R(z) = 1 - 1/z^2$, where Fatou components have a Cantor set of common boundary points. **Problem:** Say more about

- kinds 1a) and 2) in general;
- other concrete examples of kinds 2) and 3);
- analogous constructions in higher degrees.

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Matings and anti-matings in V_2 can be computed and visualized interactively with Mandel, which is available from www.mndynamics.com . A console-based implementation of slow mating for arbitrary quadratic polynomials is distributed on the arXiv with the preprint of [21].