Geometrical Approach to Inverse Scattering

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Abstract

We present a *geometrical* approach to the inverse scattering problem for the Schrödinger and Klein-Gordon equations. For given scattering operator S we show uniqueness of the potential, we give explicit limits of the high-energy behavior of the scattering operator, and we give reconstruction formulas for the potential.

Our mathematical proofs closely follow physical intuition. A key observation is that at high energies the translation of wave packets dominates over spreading during the interaction time.

1 Introduction, the Schrödinger Equation

The **Schrödinger equation** is a linear evolution equation for a function of time $t \in \mathbb{R}$ with values in a state space (phase space) $\mathcal{H}$ which is a Hilbert space:

$$\Psi(\cdot) : \mathbb{R} \to \mathcal{H}.$$

The initial value problem reads

$$i\frac{d}{dt}\,\Psi(t) = H\,\Psi(t), \quad \Psi(0) = \Psi, \tag{1.1}$$

with a linear operator H acting on $\mathcal{H}$. This type of equation includes as special cases nonrelativistic and relativistic quantum mechanics, the Dirac equation, the linear wave equation (with the usual method to transform a second order equation into a first order system), and other evolution equations. In the models mentioned above the operator H is self-adjoint on a suitably chosen domain $\mathcal{D}(H)$. Then $\exp\{-itH\}$ is a well defined unitary operator for all $t \in \mathbb{R}$ and the unique global solution of the initial value problem (1.1) is

$$\Psi(t) = e^{-itH}\Psi\,. \tag{1.2}$$

We describe our geometrical approach to the inverse problem for the Schrödinger equation as an equation which describes the motion of particles according to the laws of quantum mechanics and for the Klein-Gordon equation. The time scales for interaction and for spreading of wave functions differ at high energies. This implies the simplicity of the leading behavior of the scattering operator because only the translational part of the time evolution matters as long as the interaction is strong. We obtain explicit formulas for the high energy scattering operator which can be used to reconstruct the potential uniquely. We want to explain *why* the statements are true and how physical intuition and mathematical proofs are closely analogous.

2 Particles in Quantum Mechanics

We describe the state of a quantum mechanical particle in ν-dimensional space by a normalized vector $\Psi \in \mathcal{H}$. The vector can be represented by a square integrable function $\psi(\cdot) \in L^2(\mathbb{R}^\nu, dx)$ with volume measure dx as a function depending on the *position* $\mathbf{x} \in \mathbb{R}^\nu$, or one can use its Fourier transform

$$\hat\psi(\cdot) \in L^2(\mathbb{R}^\nu, dp), \quad \hat\psi(\mathbf{p}) := (2\pi)^{-\nu/2} \int dx\, e^{-i\mathbf{p}\mathbf{x}}\, \psi(\mathbf{x}) \tag{2.1}$$

depending on the *momentum variable* $\mathbf{p} \in \mathbb{R}^\nu$. We always assume the normalization

$$\|\Psi\|^2 = \int dx\, |\psi(\mathbf{x})|^2 = \int dp\, |\hat\psi(\mathbf{p})|^2 = 1.$$

We use for the abstract state vector a capital letter Ψ, for its representation as a function of position $\psi(\mathbf{x})$, or its momentum space wave function $\hat\psi(\mathbf{p})$, respectively, and write

$$\begin{array}{ccccc} \mathcal{H} & \longleftrightarrow & L^2(\mathbb{R}^\nu, dx) & \longleftrightarrow & L^2(\mathbb{R}^\nu, dp) \\ \Psi & \longleftrightarrow & \psi(\mathbf{x}) & \longleftrightarrow & \hat\psi(\mathbf{p}) \end{array} \tag{2.2}$$

to indicate the switching between representations.

For a given state Ψ the probability measures $\mu_\mathbf{x}$ on configuration space and $\mu_\mathbf{p}$ on momentum space, respectively,

$$\mu_\mathbf{x}(A) = \int_A dx\, |\psi(\mathbf{x})|^2 \quad \text{and} \quad \mu_\mathbf{p}(B) = \int_B dp\, |\hat\psi(\mathbf{p})|^2 \tag{2.3}$$

describe the probabilities to find the particle in $A \subset \mathbb{R}^\nu$ in configuration space or in $B \subset \mathbb{R}^\nu$ in momentum space. One may visualize such a state as a cloud of very many particles where $\mu_\mathbf{x}(A)$ describes the fraction of them which have their position in A and, similarly, $\mu_\mathbf{p}(B)$ is the fraction with momentum in B. Such a state is also called a *wave packet*.

We extend the triple of representations of state vectors to the linear operators acting on them. The Fourier transformation (2.1) interchanges differentiation and multiplication of a function with its argument. Thus we obtain for the position and momentum operators, respectively,

$$\mathbf{x} \longleftrightarrow \mathbf{x} \longleftrightarrow i\nabla_\mathbf{p}\,, \tag{2.4}$$

$$\mathbf{p} \longleftrightarrow -i\nabla_\mathbf{x} \longleftrightarrow \mathbf{p}. \tag{2.5}$$

If the forces acting on the particle are described as the negative gradient of a potential function $V(\mathbf{x})$ (conservative mechanical system) then the generator H of the time evolution, the *Hamiltonian* or *Schrödinger operator*, is the energy operator

$$H = H_0 + V(\mathbf{x}) \tag{2.6}$$

which is a sum of the kinetic energy operator H_0 – responsible for the kinematics – and the real valued potential energy which determines the dynamics.

3 Kinematics

The kinetic energy operator or *free Hamiltonian* H_0 usually is a function $H_0(\mathbf{p})$ of the momentum of the particle. We will study two typical cases, nonrelativistic (NR) and relativistic (Rel) kinematics. In the first case

$$\text{NR:} \quad H_0(\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2. \tag{3.1}$$

It acts as a multiplication operator on $\hat{\phi}$ and as a differential operator on ϕ:

$$H_0\,\Phi \quad \longleftrightarrow \quad (H_0\,\phi)(\mathbf{x}) = -\frac{1}{2m}(\Delta\phi)(\mathbf{x}) \quad \longleftrightarrow \quad H_0(\mathbf{p})\,\hat{\phi}(\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2\,\hat{\phi}(\mathbf{p}).$$

Generally, the *velocity operator* is the change of position in time:

$$\mathbf{v}(\mathbf{p}) = \frac{d}{dt}\,e^{itH_0}\,\mathbf{x}\,e^{-itH_0}\Big|_{t=0} = i\,[H_0,\,\mathbf{x}] = \nabla_{\mathbf{p}}\,H_0(\mathbf{p})\,, \tag{3.2}$$

a function of the momentum operator. In the nonrelativistic case it is unbounded:

$$\text{NR:} \quad \mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}}{m}\,. \tag{3.3}$$

Let us now turn to the scalar relativistic case:

$$\text{Rel:} \quad H_0(\mathbf{p}) = \sqrt{\mathbf{p}^2c^2 + m^2c^4} = \sqrt{\mathbf{p}^2 + m^2} \qquad (\text{speed of light } c = 1). \tag{3.4}$$

Here the velocity operator is bounded:

$$\text{Rel:} \quad \mathbf{v}(\mathbf{p}) = \nabla_{\mathbf{p}}\,H_0(\mathbf{p}) = c\,\frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2c^2 + m^2c^4}} = \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}}\,. \tag{3.5}$$

The **free time evolution** operator is a simple multiplication operator in momentum space

$$e^{-itH_0}\Phi \quad \longleftrightarrow \quad (e^{-itH_0}\phi)(\mathbf{x}) \quad \longleftrightarrow \quad e^{-itH_0(\mathbf{p})}\hat{\phi}(\mathbf{p}). \tag{3.6}$$

While for short times the free classical and quantum time evolutions differ considerably they behave similarly for large times. Asymptotically, the distribution in configuration space of a quantum state is in good approximation the same as that of the corresponding cloud of free classical particles, of the "classical wave packet". For later applications we study a particular family of states $\Phi_{\bar{\mathbf{p}}}$ with compact momentum support around a very **large "average" momentum** $\bar{\mathbf{p}} \in \mathbb{R}^\nu$. The unitary operator $\exp(i\bar{\mathbf{p}}\mathbf{x})$, a function of the position operator $\mathbf{x}$, shifts a state in momentum space by $\bar{\mathbf{p}}$:

$$\Phi_{\mathbf{0}} \qquad \longleftrightarrow \qquad \phi_{\mathbf{0}}(\cdot) \qquad \longleftrightarrow \quad \hat{\phi}_{\mathbf{0}}(\cdot) \in C_0^\infty(\mathbb{R}^\nu) \tag{3.7}$$

$$\Phi_{\bar{\mathbf{p}}} = e^{i\bar{\mathbf{p}}\mathbf{x}}\Phi_{\mathbf{0}} \longleftrightarrow \phi_{\bar{\mathbf{p}}}(\mathbf{x}) = e^{i\bar{\mathbf{p}}\mathbf{x}}\phi_{\mathbf{0}}(\mathbf{x}) \longleftrightarrow \hat{\phi}_{\bar{\mathbf{p}}}(\mathbf{p}) = \hat{\phi}_{\mathbf{0}}(\mathbf{p} - \bar{\mathbf{p}}). \tag{3.8}$$

Since $\phi_{\mathbf{0}}(\cdot) \in \mathcal{S}(\mathbb{R}^\nu)$, the Schwartz space of rapidly decreasing functions, these states are well localized in configuration space, too, uniformly in $\bar{\mathbf{p}}$. They have average velocities around $\mathbf{v}(\bar{\mathbf{p}}) \in \mathbb{R}^\nu$, where

$$\mathbf{v}(\bar{\mathbf{p}}) = \nabla H_0(\bar{\mathbf{p}}) =: v(\bar{\mathbf{p}})\,\boldsymbol{\omega} = \begin{cases} \bar{\mathbf{p}}/m & \text{NR}\,, \\ \bar{\mathbf{p}}/\sqrt{\bar{\mathbf{p}}^2 + m^2} & \text{Rel}\,, \end{cases} \qquad \boldsymbol{\omega} = \frac{\mathbf{v}(\bar{\mathbf{p}})}{|\mathbf{v}(\bar{\mathbf{p}})|} \parallel \bar{\mathbf{p}}. \tag{3.9}$$

In our context we have to control the localization in configuration space of freely evolving wave packets. This depends mainly on the support of the state in velocity (momentum) space. Therefore, we have chosen compactly supported momentum space wave functions. Then in configuration space the states cannot have compact support as well but rapid falloff is sufficient there. A special case of such **non–propagation properties** of quantum wave packets for long times is

$$\int\limits_{|\mathbf{x}|<t\,v(\bar{\mathbf{p}})/2} dx\,|(e^{-itH_0}\phi_{\bar{\mathbf{p}}})(\mathbf{x})|^2 < \frac{\mathrm{const}(\Phi_{\mathbf{0}},\,n)}{(1+|t\,v(\bar{\mathbf{p}})|)^n} \tag{3.10}$$

for any $n \in \mathbb{N}$ *uniformly* for large $\bar{\mathbf{p}}$. A classical free particle which starts at time 0 from the origin and has momentum $\mathbf{p} \in \operatorname{supp}\hat{\phi}_{\bar{\mathbf{p}}}$ will be localized at time t in the region

$$\mathbf{x}(t) \in \left\{\mathbf{x} = t\,\mathbf{v}(\mathbf{p}) \;\middle|\; \mathbf{p} \in \operatorname{supp}\hat{\phi}_{\bar{\mathbf{p}}}\right\} \subset \{\,\mathbf{x} \;|\; |\mathbf{x} - t\,\mathbf{v}(\bar{\mathbf{p}})| < v(\bar{\mathbf{p}})/3\}\,. \tag{3.11}$$

The “classically forbidden” region $|\mathbf{x}| < t\,v(\bar{\mathbf{p}})/2$ is separated from the “allowed region” by at least $t\,v(\bar{\mathbf{p}})/6$. The state mainly propagates within the classically allowed region which moves away from the origin with a positive minimal speed. The “quantum tails” of the wave packet in the classically forbidden region do not vanish, nevertheless, they decay very fast in time, both in the future and past. This is physically and mathematically in close analogy to rays versus waves in optics. While the shadow behind an obstacle is not totally black due to diffraction it is, nevertheless, quite dark away from the region which can be reached by straight rays (the role of the increasing separation $t\,v(\bar{\mathbf{p}})/6$).

4 Dynamics

The interacting (perturbed) time evolution is generated by the Hamiltonian H,

$$e^{-itH}\Psi, \quad H = H_0 + V(\mathbf{x}). \tag{4.1}$$

We will consider here *short-range* potentials $V(\mathbf{x})$ which are roughly those which decrease at least like $|\mathbf{x}|^{-(1+\varepsilon)}$, $\varepsilon > 0$, as $|\mathbf{x}| \to \infty$. More precisely, the set of short-range potentials is

$$\mathcal{V}^s = \left\{V \;\middle|\; \int_0^\infty \sup_{|\mathbf{x}|\geq R} |V(\mathbf{x})| \;\, dR < \infty\right\}. \tag{4.2}$$

For simplicity of presentation we will restrict ourselves in this paper to bounded potentials. Singular and long-range potentials can be included using standard techniques.

In the present context a short-range potential behaves similarly to a compactly supported one. Depending on the required accuracy it is essentially concentrated in a ball of some radius R around the origin.

The influence on the particle by the force $-\nabla V(\mathbf{x})$ is relevant only as long as the particle is essentially localized in the interaction region, i.e. where the potential is strong. We study the scattering states which form the *continuous* spectral subspace $\mathcal{H}^{\mathrm{cont}}(H) = \{\text{eigenvectors of } H\}^{\perp}$, they leave the interaction region for large times.

5 Scattering

For short-range potentials the asymptotic motion of scattering states is an essentially free motion: For any scattering state $\Psi \in \mathcal{H}^{\mathrm{cont}}(H)$ there exist free asymptotic configurations $\Phi^{\pm} \in \mathcal{H}$ such that

$$\left\| e^{-it\,[H_0+V]}\,\Psi - e^{-itH_0}\,\Phi^{\pm} \right\| \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{5.1}$$

This is usually called asymptotic completeness of the wave operators. Similarly, for any *incoming* configuration Φ^- or *outgoing* Φ^+ there is a corresponding state $\Psi \in \mathcal{H}^{\mathrm{cont}}(H)$ such that (5.1) holds (existence of wave operators).

A convenient tool to describe scattering is the scattering operator S which maps an incoming configuration Φ^- to the corresponding outgoing configuration Φ^+ of the same state Ψ. For given Φ^- let

$$\Psi = \lim_{t_- \to -\infty} e^{i\,t_-[H_0+V]}\; e^{-i\,t_-\,H_0}\; \Phi^- \qquad \text{and}$$

$$\Phi^+ = \lim_{t_+ \to \infty} e^{i\,t_+ H_0}\; e^{-i\,t_+[H_0+V]}\; \Psi.$$

Then

$$S(t_+, t_-) := e^{i\,t_+ H_0}\; e^{-i\,t_+[H_0+V]}\; e^{i\,t_-[H_0+V]}\; e^{-i\,t_-\,H_0}\;, \tag{5.2}$$

$$S := \operatorname*{s-lim}_{\substack{t_+ \to \infty \\ t_- \to -\infty}} \; S(t_+, t_-), \quad \text{satisfies} \quad S\,\Phi^- = \Phi^+. \tag{5.3}$$

For microscopic particles for which quantum mechanics is an adequate description one cannot really observe more details of the scattering process than those encoded in the scattering operator. We denote the mapping

$$\mathcal{V}^s \to L(\mathcal{H}), \quad V \mapsto S = S(V) \tag{5.4}$$

as the *scattering map* from short-range potentials to bounded (unitary) scattering operators on the Hilbert space of asymptotic configurations.

The **direct** problem of scattering theory is to determine for a given potential V the scattering operator while the **inverse** problem is to determine the potential(s) if the scattering operator or part of it is known.

6 Uniqueness of the Potential

We denote by $F(H_0 \geq E)$ the multiplication operator in momentum space with the characteristic function of the set $\{\mathbf{p} \in \mathbb{R}^\nu \mid H_0(\mathbf{p}) \geq E\}$, i.e. the spectral projection of the kinetic energy operator to energies above E. The main results about uniqueness are of the following form. They are a corollary of the asymptotic behavior of the scattering operator shown below.

Theorem 6.1 *The scattering map $S\colon \mathcal{V}^s \to L(\mathcal{H})$ is injective. Actually, the high-energy part of the scattering operator alone: $S\,F(H_0 \geq E)$, E arbitrarily large, determines the short-range potential uniquely.*

7 Time Scales and Length Scales for Interaction and Spreading

For high energy states as constructed in (3.8) scattering theory becomes simple because two time scales, an interaction time $T_I(\bar{\mathbf{p}})$ and a kinematical time of spreading $T_{Sp}(\bar{\mathbf{p}})$ satisfy $T_I(\bar{\mathbf{p}})/T_{Sp}(\bar{\mathbf{p}}) \to 0$ as $|\bar{\mathbf{p}}| \to \infty$. For a potential which is essentially supported in a ball of radius R the **interaction time** is of the order $T_I(\bar{\mathbf{p}}) = R/v(\bar{\mathbf{p}})$. More precisely, for $\Phi_{\bar{\mathbf{p}}}$, $\Phi'_{\bar{\mathbf{p}}}$ as in (3.8) and any $\varepsilon > 0$ there is a radius $\rho(\varepsilon)$ such that uniformly for large $|\bar{\mathbf{p}}|$

$$|(\Phi'_{\bar{\mathbf{p}}},\ [\,S - S(t_+, t_-)\,]\ \Phi_{\bar{\mathbf{p}}})| < \frac{\varepsilon}{v(\bar{\mathbf{p}})} \quad \text{if} \quad \pm t_\pm > \rho(\varepsilon)/v(\bar{\mathbf{p}}) \approx T_I(\bar{\mathbf{p}}). \tag{7.1}$$

$\rho(\varepsilon)$ is the length scale L_I of interaction which is independent of $\bar{\mathbf{p}}$. Intuitively, the radius of the interaction region and the extension in configuration space of the states up to effects of size ε sum up to $\rho(\varepsilon)$. The interaction time (and consequently the interaction strength) decreases with $|\bar{\mathbf{p}}| \to \infty$ in the nonrelativistic case and remains fixed and positive for relativistic kinematics.

The kinematical **time scale of spreading** $T_{Sp}(\bar{\mathbf{p}})$ denotes the time after which spreading of wave packets becomes relevant in the time evolution. As

$$H_0(\mathbf{p})\,\Psi_{\bar{\mathbf{p}}} = H_0(\mathbf{p})\,e^{i\bar{\mathbf{p}}\mathbf{x}}\,\Psi_{\mathbf{0}} = e^{i\bar{\mathbf{p}}\mathbf{x}}\,H_0(\bar{\mathbf{p}}+\mathbf{p})\,\Psi_{\mathbf{0}}$$

we will expand the kinetic energy function around $\bar{\mathbf{p}}$

$$H_0(\bar{\mathbf{p}}+\mathbf{p}) =: H_0(\bar{\mathbf{p}}) + \nabla H_0(\bar{\mathbf{p}})\cdot\mathbf{p} + H_2(\bar{\mathbf{p}}, \mathbf{p}). \tag{7.2}$$

The first summand is a number giving an irrelevant phase, the second equals $\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}$ by (3.9). It is the dominant term which – as a multiple of the momentum operator – generates a translation of the wave packet *without changing its shape*. Only the third term H_2 (which is defined by (7.2)) is responsible for the spreading of the wave packet. In our examples of "power like" Hamiltonians this part of the Hamiltonian is weak compared to the translational component: On a compact subset of momentum space like $\mathbf{p} \in \operatorname{supp}\hat{\phi}_{\mathbf{0}}$

$$\frac{T_I(\bar{\mathbf{p}})}{T_{Sp}(\bar{\mathbf{p}})} \sim \frac{|H_2(\bar{\mathbf{p}},\ \mathbf{p})|}{v(\bar{\mathbf{p}})} \le \frac{\text{const}}{|\bar{\mathbf{p}}|} \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} 0. \tag{7.3}$$

Therefore, the time $T_{Sp}(\bar{\mathbf{p}})$ is by a factor proportional to $|\bar{\mathbf{p}}|$ longer than $T_I(\bar{\mathbf{p}})$, *independent* of the kinematics. For large $|\bar{\mathbf{p}}|$ we may choose times when the scattering due to the potential is over but the spreading has not yet really started. Alternatively, the time $T_{Sp}(\bar{\mathbf{p}})$ translates into a length scale $L_{Sp}(\bar{\mathbf{p}}) = v(\bar{\mathbf{p}})\,T_{Sp}(\bar{\mathbf{p}})$. A particle has to travel at least that far until spreading may become visible. $L_{Sp}(\bar{\mathbf{p}})$ increases proportional to $|\bar{\mathbf{p}}|$ for both kinematics. Again, the ratio $L_I/L_{Sp} = T_I/T_{Sp} \sim 1/|\bar{\mathbf{p}}| \to 0$ for any precision ε.

Usually, an interacting time evolution is complicated because the translation of a wave packet, its spreading, and the influence of the potential all occur at the same time and in the same region. In the high-energy limit it is sufficient for the calculation of the scattering operator to treat translation of wave packets rather than their correct free evolution. Since in this limit spreading occurs only when and where the interaction is negligible, i.e. when the free and interacting time evolutions are almost the same, the effect of spreading is canceled (becomes invisible) in the scattering operator. Thus, high energy scattering is simple and it can be inverted simply!

8 High Energy Scattering

The crucial uniformity of the estimate (7.1) enables us to interchange the limits $\pm t_\pm \to \infty$ and $|\bar{\mathbf{p}}| \to \infty$. This simplifies the remaining discussion very much. Actually, not the time but the separation from the region of a strong potential determines the quality of approximation. With correspondingly chosen variables $r_\pm := t_\pm \, v(\bar{\mathbf{p}})$ we get

$$\begin{aligned}
&\lim_{|\bar{\mathbf{p}}|\to\infty} (\Phi'_{\bar{\mathbf{p}}},\ S\,\Phi_{\bar{\mathbf{p}}}) = \lim_{|\bar{\mathbf{p}}|\to\infty}\ \lim_{\pm t_\pm\to\infty} (\Phi'_{\bar{\mathbf{p}}},\ S(t_+, t_-)\,\Phi_{\bar{\mathbf{p}}}) \\
&= \lim_{|\bar{\mathbf{p}}|\to\infty}\ \lim_{\pm r_\pm\to\infty} \left(\Phi'_{\bar{\mathbf{p}}},\ S\left(\frac{r_+}{v(\bar{\mathbf{p}})}, \frac{r_-}{v(\bar{\mathbf{p}})}\right)\Phi_{\bar{\mathbf{p}}}\right) \\
&= \lim_{\pm r_\pm\to\infty}\ \lim_{|\bar{\mathbf{p}}|\to\infty} \left(\Phi'_{\bar{\mathbf{p}}},\ S\left(\frac{r_+}{v(\bar{\mathbf{p}})}, \frac{r_-}{v(\bar{\mathbf{p}})}\right)\Phi_{\bar{\mathbf{p}}}\right). \qquad (8.1)
\end{aligned}$$

As seen in (7.1) the asymptotic equality (8.1) remains true even after multiplication with $v(\bar{\mathbf{p}})$ which is a much stronger statement in the nonrelativistic case. To determine

$$(\Phi'_{\bar{\mathbf{p}}},\ S(t_+, t_-)\,\Phi_{\bar{\mathbf{p}}}) = \left(\Phi'_{\mathbf{0}},\ e^{-i\bar{\mathbf{p}}\mathbf{x}}\,S(t_+, t_-)\,e^{i\bar{\mathbf{p}}\mathbf{x}}\,\Phi_{\mathbf{0}}\right) \qquad (8.2)$$

for large finite times and $\bar{\mathbf{p}}$ consider e.g. the second pair of factors in (5.2).

$$\begin{aligned}
&e^{-i\bar{\mathbf{p}}\mathbf{x}}\ e^{it_-[H_0+V]}\ e^{-it_- H_0}\ e^{i\bar{\mathbf{p}}\mathbf{x}} \\
&= e^{it_-[H_0(\mathbf{p}+\bar{\mathbf{p}})+V(\mathbf{x})]}\ e^{-it_- H_0(\mathbf{p}+\bar{\mathbf{p}})} \\
&= e^{it_-[H_0(\bar{\mathbf{p}})+\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})+V(\mathbf{x})]}\ e^{-it_-[H_0(\bar{\mathbf{p}})+\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})]} \\
&= e^{it_-[\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})+V(\mathbf{x})]}\ e^{-it_-[\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})]} \\
&= e^{ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}+\{V(\mathbf{x})/v(\bar{\mathbf{p}})\}]}\ e^{-ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}]} \qquad (8.3)
\end{aligned}$$

using again $t_\pm = r_\pm/v(\bar{\mathbf{p}})$ and the direction $\boldsymbol{\omega} = \mathbf{v}(\bar{\mathbf{p}})/v(\bar{\mathbf{p}})$ as in (3.9). Due to (7.3) the functions of the momentum operator

$$[\,\boldsymbol{\omega}\cdot\mathbf{p} + \{H_2(\bar{\mathbf{p}}, \mathbf{p})/v(\bar{\mathbf{p}})\}\,] \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \boldsymbol{\omega}\cdot\mathbf{p} \qquad (8.4)$$

converge in strong resolvent sense and similarly for the other exponent. Therefore, for fixed r_- and large $|\bar{\mathbf{p}}|$ the following approximation is good:

$$\begin{aligned}
&e^{ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}+\{V(\mathbf{x})/v(\bar{\mathbf{p}})\}]}\ e^{-ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}]} \\
&\approx e^{ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{V(\mathbf{x})/v(\bar{\mathbf{p}})\}]}\ e^{-ir_-\boldsymbol{\omega}\cdot\mathbf{p}} \qquad (8.5) \\
&= \exp\left\{\frac{-i}{v(\bar{\mathbf{p}})}\int_{r_-}^{0} dr\, V(\mathbf{x}+\boldsymbol{\omega}\,r)\right\}. \qquad (8.6)
\end{aligned}$$

The approximation (8.5) is the only approximation we have to make! If $\{H_2(\bar{\mathbf{p}}, \mathbf{p})/v(\bar{\mathbf{p}})\}$ would commute with $\{V(\mathbf{x})/v(\bar{\mathbf{p}})\}$ then we would have exact cancellation and (8.5) would be an equality as well. A careful estimate of the correction terms can be given for all Hamiltonians considered here. It is uniform in r_- and when compared to $\{V(\mathbf{x})/v(\bar{\mathbf{p}})\}$ it has additional falloff like $1/|\bar{\mathbf{p}}|$ for $\bar{\mathbf{p}} \to \infty$ due to (7.3).

Combining (8.6) with the corresponding term for positive times we obtain for large $|\bar{\mathbf{p}}|$

$$(\Phi'_{\bar{\mathbf{p}}}\,,\ S\Phi_{\bar{\mathbf{p}}}) \approx \left(\Phi'_{\mathbf{0}}\,,\ \exp\left\{\frac{-i}{v(\bar{\mathbf{p}})}\int_{-\infty}^{\infty} dr\ V(\mathbf{x}+\boldsymbol{\omega}r)\right\}\ \Phi_{\mathbf{0}}\right). \tag{8.7}$$

9 High Energy Limits of the Scattering Operator

Next we give the limiting behavior of the scattering operator in simple cases, $\Phi_{\bar{\mathbf{p}}}$, $\Phi'_{\bar{\mathbf{p}}}$, and $\bar{\mathbf{p}} \in \mathbb{R}^\nu$ as given in (3.8). The strong influence of the kinematics is clearly visible. For an overview of many further results see [5] and the references.

Theorem 9.1 (scalar relativistic, short-range, [15])
For the scalar relativistic Hamiltonian

$$H = \sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{x})$$

with $v(\bar{\mathbf{p}}) \to 1$ *one obtains*

$$(\Phi'_{\bar{\mathbf{p}}},\ S\,\Phi_{\bar{\mathbf{p}}}) \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \left(\Phi'_{\mathbf{0}},\ \exp\left\{-i\int dr\ V(\mathbf{x}+\boldsymbol{\omega}r)\right\}\ \Phi_{\mathbf{0}}\right). \tag{9.1}$$

If, however, $v(\bar{\mathbf{p}}) \to \infty$ we can expand the exponential in (8.7)

$$\begin{aligned} &\exp\left\{-\frac{i}{v(\bar{\mathbf{p}})}\int dr\ V(\mathbf{x}+\boldsymbol{\omega}r)\right\} \\ &\quad\approx 1 - \frac{i}{v(\bar{\mathbf{p}})}\int dr\ V(\mathbf{x}+\boldsymbol{\omega}r) + \cdots \end{aligned} \tag{9.2}$$

which explains the following nonrelativistic result. The leading behavior of the scattering operator is the identity operator (no scattering). The next order correction depends on the potential.

Theorem 9.2 (nonrelativistic, short-range, [10], [4], [18], [6], [8])
For the Hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x})$$

$$v(\bar{\mathbf{p}})\ (\Phi'_{\bar{\mathbf{p}}},\ i(S-\mathbf{1})\Phi_{\bar{\mathbf{p}}}) \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \int dr\ (\Phi'_{\mathbf{0}},\ V(\mathbf{x}+\boldsymbol{\omega}r)\Phi_{\mathbf{0}}). \tag{9.3}$$

In the quotations we have included similar results obtained by other methods, sometimes under more restrictive assumptions. This result is to be expected from the Born approximation. It holds also under the given weaker assumptions on the falloff of the potential where the validity of the Born approximation is not established.

The estimate (7.1) and the remark following (8.6) justify that multiplication with $v(\bar{\mathbf{p}}) \sim |\bar{\mathbf{p}}|$ is permitted. The terms omitted in the approximation are smaller than those involving $V/v(\bar{\mathbf{p}})$.

Remark

In all these limits there are **error bounds** for large but finite $|\bar{\mathbf{p}}|$ which are explicit. E.g. in equation (9.3) we obtain

$$\left| v(\bar{\mathbf{p}})\, (\Phi'_{\bar{\mathbf{p}}},\ i(S-\mathbf{1})\, \Phi_{\bar{\mathbf{p}}}) - \int dr\, (\Phi'_{\mathbf{0}},\, V(\mathbf{x}+\boldsymbol{\omega} r)\, \Phi_{\mathbf{0}}) \right| \leq \frac{\mathrm{const}(\Phi'_{\mathbf{0}},\, \Phi_{\mathbf{0}},\, V)}{|\bar{\mathbf{p}}|} .$$

10 Reconstruction of the Potential

The condition $\nu \geq 2$ (*multi*dimensional inverse problem) enters here to obtain from the above limits reconstruction formulas and uniqueness. For bounded continuous (or more general) functions V the expression

$$X(\mathbf{x}, \boldsymbol{\omega}) := \int dr\, V(\mathbf{x}+\boldsymbol{\omega} r) \tag{10.1}$$

is the X-ray transform of V. In $\nu = 2$ dimensions lines and hyperplanes are the same. Therefore, (10.1) is the Radon transform as well. The latter is known to be uniquely invertible because the assumption (4.2) implies $V \in L^2(\mathbb{R}^2)$, see e.g. Theorem 2.17 in Chapter I of [11]. The inverse Radon transform yields the unique potential. In higher dimensions one fixes e.g. $x_3, \ldots, x_\nu$ and reconstructs the "slices" subsequently. In particular, it is sufficient to vary $\boldsymbol{\omega}$ in a two dimensional plane. For unbounded or discontinuous potentials the expectation value between states from a dense set of nice vectors (like those which satisfy (3.7)) effectively smoothes the potential. This is enough to reconstruct the potential as a multiplication operator.

11 The Klein-Gordon Equation

The Klein-Gordon equation describes the evolution of a wave-packet for a relativistic spin-0 particle of mass $m > 0$ in $\mathbb{R}^\nu$. Setting the velocity of light $c = 1$, Planck's constant $\hbar = 1$ and the charge $q = 1$, we have the free equation

$$\ddot{u} = \Delta u - m^2 u\, , \quad \text{or} \quad \ddot{u} + \left[\mathbf{p}^2 + m^2\right] u = 0 \tag{11.1}$$

with the momentum operator $\mathbf{p} = -i\nabla$. For a particle in an electromagnetic field $\mathbf{E} = -\nabla A_0$, the corresponding equation reads

$$\begin{gathered} (\partial_t + iA_0(\mathbf{x}))^2 u = \Delta u - m^2 u\, , \qquad \text{or} \\ \ddot{u} + i2A_0(\mathbf{x})\, \dot{u} + \left[\mathbf{p}^2 + m^2 - A_0(\mathbf{x})^2\right] u = 0\, , \end{gathered} \tag{11.2}$$

thus $A_0 : \mathbb{R}^\nu \to \mathbb{R}$ influences the evolution of $u(t) : \mathbb{R}^\nu \to \mathbb{C}$. In the direct scattering problem, the large time/large distance asymptotics of solutions of (11.2) are described by a scattering operator S, that is determined from a suitably decaying potential A_0. We shall solve the inverse problem: Determine A_0 from S, thus from data that are in principle measurable in a scattering experiment. All spin-0 particles in nature are unstable, and the one-particle Klein-Gordon equation has very limited physical applications. We believe that it is interesting nevertheless, since we can compare the results to the Dirac equation (a relativistic wave equation for spin-1/2 particles), and to acoustic scattering.

To obtain a system of first-order equations, we set

$$\tilde\psi(\mathbf{x}) = \begin{pmatrix} \tilde\psi_1(\mathbf{x}) \\ \tilde\psi_2(\mathbf{x}) \end{pmatrix} := \begin{pmatrix} u(\mathbf{x}) \\ \dot u(\mathbf{x}) \end{pmatrix} , \qquad \text{and} \qquad \hat{\tilde\psi}(\mathbf{p}) = \begin{pmatrix} \hat{\tilde\psi}_1(\mathbf{p}) \\ \hat{\tilde\psi}_2(\mathbf{p}) \end{pmatrix} \tag{11.3}$$

for the momentum representation. The tilde is used because we will soon introduce another representation, where the tilde is omitted. Now (11.1) $\Leftrightarrow i\dot{\tilde\Psi} = \tilde H_0\, \tilde\Psi$ with $\tilde H_0 = \begin{pmatrix} 0 & i \\ -i\, B_0^2 & 0 \end{pmatrix}$ and $B_0^2 = -\Delta + m^2 = \mathbf{p}^2 + m^2$, and (11.2) $\Leftrightarrow i\dot{\tilde\Psi} = \tilde H_1\, \tilde\Psi$ with $\tilde H_1 = \begin{pmatrix} 0 & i \\ -i\, B_1^2 & 2A_0(\mathbf{x}) \end{pmatrix}$, where $B_1^2 = -\Delta + m^2 - A_0(\mathbf{x})^2 = \mathbf{p}^2 + m^2 - A_0(\mathbf{x})^2$. We have to specify Hilbert spaces and domains for $\tilde H_0$ and $\tilde H_1$, such that these operators are well-defined and self-adjoint. The choice of $\tilde{\mathcal{H}} = L^2(\mathbb{R}^\nu,\, \mathbb{C}^2)$ is not possible, and before we can define the correct spaces, we shall take a look at B_0^2 and B_1^2 in $\mathcal{H} = L^2(\mathbb{R}^\nu,\, \mathbb{C})$: B_0^2 is self-adjoint and strictly positive on its domain $H^2(\mathbb{R}^\nu)$ (a Sobolev space), and we assume $A_0 \in L^\infty(\mathbb{R}^\nu,\, \mathbb{R})$ with $\|A_0^2\Phi\|_{L^2} \le a\|B_0^2\Phi\|_{L^2}$ for some $a < 1$ and all $\Phi \in H^2$. By the Kato-Rellich Theorem, B_1^2 is self-adjoint and strictly positive on $H^2(\mathbb{R}^\nu)$. Now $B_k := \sqrt{B_k^2}$ is a well-defined self-adjoint operator on $H^1(\mathbb{R}^\nu)$. B_0^2 and B_1^2 are second-order differential operators, and B_0 is a pseudo-differential operator: $\widehat{(B_0\Phi)}(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}\,\hat\Phi(\mathbf{p})$. There is no explicit expression for B_1. If we define the Hilbert spaces

$$\tilde{\mathcal{H}}_0 := H^1(\mathbb{R}^\nu) \oplus L^2(\mathbb{R}^\nu) \quad \text{with} \quad \|\tilde\psi\|^2_{\tilde{\mathcal{H}}_0} = \|B_0\,\tilde\psi_1\|^2_{L^2} + \|\tilde\psi_2\|^2_{L^2} \quad \text{and} \tag{11.4}$$

$$\tilde{\mathcal{H}}_1 := H^1(\mathbb{R}^\nu) \oplus L^2(\mathbb{R}^\nu) \quad \text{with} \quad \|\tilde\psi\|^2_{\tilde{\mathcal{H}}_1} = \|B_1\,\tilde\psi_1\|^2_{L^2} + \|\tilde\psi_2\|^2_{L^2} , \tag{11.5}$$

then $\tilde H_k$ is self-adjoint in $\tilde{\mathcal{H}}_k$ with $D_{\tilde H_k} = H^2 \oplus H^1$. Now $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_1$ are equal as sets, they have different but equivalent norms, and the natural identification operator $J : \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_1$ is a linear isomorphism. For the Schrödinger- or Dirac equation, the integrand $|\psi(\mathbf{x})|^2$ of the squared norm is interpreted as a probability or charge density, and for the Klein-Gordon equation, $\tilde\psi_1^*(\mathbf{x})(-\Delta + m^2 - A_0^2(\mathbf{x}))\tilde\psi_1(\mathbf{x}) + |\tilde\psi_2(\mathbf{x})|^2$ represents an energy density.

Now $\tilde H_0$ and $\tilde H_1$ act on different Hilbert spaces, and the definition of the wave operators must be modified: The identification operator J is used to compare the interacting states with free asymptotic configurations. $\mathrm{e}^{-i\tilde H_1 t}\tilde\Psi_\pm - J\,\mathrm{e}^{-i\tilde H_o t}\tilde\Psi \to 0$ for $t \to \pm\infty$ leads to $\tilde\Psi_\pm = \tilde\Omega_\pm\tilde\Psi$ with the wave operators

$$\tilde\Omega_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{i\tilde H_1 t} J\,\mathrm{e}^{-i\tilde H_0 t} \;:\; \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_1 \,. \tag{11.6}$$

J is not isometric, but the unitary operator $\tilde T = \begin{pmatrix} \frac{1}{B_1} B_0 & 0 \\ 0 & 1 \end{pmatrix}$ "behaves like J for large $|\mathbf{x}|$", thus $\tilde\Omega_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{i\tilde H_1 t}\tilde T\,\mathrm{e}^{-i\tilde H_0 t}$, and it is isometric. On suitable states $\tilde\Psi$ with momentum support bounded away from the origin, we have the representation

$$\tilde\Omega_\pm\,\tilde\Psi = \tilde\Psi + i\int_0^{\pm\infty} dt\; \mathrm{e}^{i\tilde H_1 t}(\tilde H_1 J - J\tilde H_0)\,\mathrm{e}^{-i\tilde H_0 t}\,\tilde\Psi \tag{11.7}$$

as an absolutely convergent Riemann or Bochner integral. It is obtained by writing the RHS of (11.6) as an integral of its derivative.

We shall introduce the Foldy-Wouthuysen representation of $\tilde{\mathcal{H}}_k$: For $\tilde{\Psi} \in \tilde{\mathcal{H}}_0$ or $\tilde{\Psi} \in \tilde{\mathcal{H}}_1$ set $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} := \begin{pmatrix} B_0 \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix}$. In the FW-momentum representation, we have $\hat{\psi}(\mathbf{p}) = \begin{pmatrix} \sqrt{\mathbf{p}^2 + m^2}\ \hat{\tilde{\psi}}_1(\mathbf{p}) \\ \hat{\tilde{\psi}}_2(\mathbf{p}) \end{pmatrix}$, and the FW-position representation is given by the inverse Fourier transform $\psi(\mathbf{y})$ of $\hat{\psi}(\mathbf{p})$. The Newton-Wigner position operator $\mathbf{y}$ is defined as multiplication with $\mathbf{y}$ in the representation $\psi(\mathbf{y})$. Now $\mathcal{H}_0 = L^2(\mathbb{R}^\nu) \oplus L^2(\mathbb{R}^\nu) = \mathcal{H}_1$ as sets, and we keep the notation J for the natural identification operator. The inner products are given by

$$\|\psi\|^2_{\mathcal{H}_0} = \|\psi_1\|^2_{L^2} + \|\psi_2\|^2_{L^2} \qquad \|\psi\|^2_{\mathcal{H}_1} = \|B_1 \frac{1}{B_0} \psi_1\|^2_{L^2} + \|\psi_2\|^2_{L^2} . \tag{11.8}$$

(B_0 and B_1 are isomorphisms $L^2 \to H^1$, thus $B_1\, 1/B_0$ is an isomorphism $L^2 \to L^2$.) The Foldy-Wouthuysen representation of the Hamiltonians is

$$H_0 = \begin{pmatrix} 0 & i\sqrt{\mathbf{p}^2 + m^2} \\ -i\sqrt{\mathbf{p}^2 + m^2} & 0 \end{pmatrix} \qquad \text{and} \tag{11.9}$$

$$H_1 = \begin{pmatrix} 0 & i\sqrt{\mathbf{p}^2 + m^2} \\ -i\,[\mathbf{p}^2 + m^2 - A_0^2(\mathbf{y})]\, \frac{1}{\sqrt{\mathbf{p}^2+m^2}} & 2A_0(\mathbf{y}) \end{pmatrix} \tag{11.10}$$

The inner product of $\mathcal{H}_1$ can be written as $(\Phi,\, \Psi)_{\mathcal{H}_1} = (J^{-1}\Phi,\, g\, J^{-1}\Psi)_{\mathcal{H}_0}$ for $\Phi,\, \Psi \in \mathcal{H}_1$, where $g := \begin{pmatrix} 1 - \frac{1}{\sqrt{\mathbf{p}^2+m^2}} A_0^2(\mathbf{y}) \frac{1}{\sqrt{\mathbf{p}^2+m^2}} & 0 \\ 0 & 1 \end{pmatrix}$ is a strictly positive, bounded self-adjoint operator on $\mathcal{H}_0$. The S-matrix is given by

$$S = \Omega_+^* \,\Omega_- = (J^{-1}\Omega_+)^* g\, (J^{-1}\Omega_-) . \tag{11.11}$$

In contrast to $\mathbf{x}$, the Newton-Wigner position operator $\mathbf{y}$ is self-adjoint. We have $H_0 = \sqrt{\mathbf{p}^2 + m^2}\, \beta$ with the matrix $\beta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. The velocity is given by

$$\mathbf{v} = i[H_0,\, \mathbf{y}] = \nabla_{\mathbf{p}} H_0 = \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}}\, \beta . \tag{11.12}$$

The eigenspaces $\beta = \pm 1$ are the spectral subspaces of positive/negative kinetic energy. The negative energy subspace corresponds to anti-particles, here we have $\mathbf{v} = -\dfrac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}}$.

12 Inverse Scattering for the Klein-Gordon Equation

The NW-position operator generates translations in momentum space:

$$\Psi_{\bar{\mathbf{p}}} := \mathrm{e}^{i\bar{\mathbf{p}}\cdot\mathbf{y}}\, \Psi_0 \quad \longleftrightarrow \quad \psi_{\bar{\mathbf{p}}}(\mathbf{y}) = \mathrm{e}^{i\bar{\mathbf{p}}\cdot\mathbf{y}}\, \psi_0(\mathbf{y}) \quad \longleftrightarrow \quad \hat{\psi}_{\bar{\mathbf{p}}}(\mathbf{p}) = \hat{\psi}_0(\mathbf{p}-\bar{\mathbf{p}}) \tag{12.1}$$

with $\bar{\mathbf{p}} = \bar{p}\boldsymbol{\omega}$, $\boldsymbol{\omega} \in S^{\nu-1}$, $\bar{p} \geq 0$. We shall consider the high-energy asymptotics of scattering by letting $\bar{p} \to \infty$. Now

$$\mathrm{e}^{-i\bar{\mathbf{p}}\cdot\mathbf{y}} \left(S\,\psi_{\bar{\mathbf{p}}} \right) = \left(\mathrm{e}^{-i\bar{\mathbf{p}}\cdot\mathbf{y}}\, S\, \mathrm{e}^{i\bar{\mathbf{p}}\cdot\mathbf{y}} \right) \Psi_0 \,, \tag{12.2}$$

and we have

Theorem 12.1 *Suppose that $\nu \in \mathbb{N}$, $m > 0$ and S is the scattering operator for a Klein-Gordon particle of mass m in an electrostatic field $\mathbf{E} = \nabla A_0$, where $A_0 : \mathbb{R}^\nu \to \mathbb{R}$ is continuous and vanishes at infinity with integrable decay: $\int_0^\infty dR\, \|\chi(|\mathbf{x}| > R)\, A_0(\mathbf{x})\|_\infty < \infty$. Moreover, we make the Kato-Rellich assumption $\|A_0^2\Psi\| \leq a\|(\mathbf{p}^2 + m^2)\Psi\|$ with $a < 1$. Then the high-energy asymptotics of S are given by*

$$\underset{\bar{p}\to\infty}{\text{s-lim}}\ \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\, S\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \exp\left\{ -i \int_{-\infty}^{\infty} dr\, A_0(\mathbf{y}+r\boldsymbol{\omega}) \right\}, \tag{12.3}$$

where $\bar{\mathbf{p}} = \bar{p}\boldsymbol{\omega}$ with $\boldsymbol{\omega} \in S^{\nu-1}$. If $\nu \geq 2$, then A_0 can be reconstructed uniquely from the scattering operator S.

Existence of the wave operators and completeness (i.e. $\mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+)$) can be shown with standard techniques. By (11.11) we have

$$\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\, S\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \left(\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} (J^{-1}\Omega_+) \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} \right)^* \left(\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\, g\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} \right) \left(\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} (J^{-1}\Omega_-) \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} \right).$$

The term in the middle is negligible for $\bar{p} \to \infty$ due to the strong convergence

$$\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\, g\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \begin{pmatrix} 1 - \frac{1}{\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2+m^2}} A_0^2(\mathbf{y}) \frac{1}{\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2+m^2}} & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Lemma 12.2 below, we have

$$\underset{\bar{p}\to\infty}{\text{s-lim}}\ \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\, J^{-1}\Omega_\pm \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \exp\left\{ i \int_0^{\pm\infty} dr\, W(r) \right\},$$

where $W(r)$ is the multiplication operator in the Foldy-Wouthuysen position representation $\psi(\mathbf{y})$ given by

$$\begin{aligned} W(r) &= \mathrm{e}^{i\beta\boldsymbol{\omega}\cdot\mathbf{p}r}\, A_0(\mathbf{y})\, \mathrm{e}^{-i\beta\boldsymbol{\omega}\cdot\mathbf{p}r} \\ &= \mathrm{e}^{i\boldsymbol{\omega}\cdot\mathbf{p}r}\, \frac{1+\beta}{2}\, A_0(\mathbf{y})\, \mathrm{e}^{-i\boldsymbol{\omega}\cdot\mathbf{p}r} + \mathrm{e}^{-i\boldsymbol{\omega}\cdot\mathbf{p}r}\, \frac{1-\beta}{2}\, A_0(\mathbf{y})\, \mathrm{e}^{i\boldsymbol{\omega}\cdot\mathbf{p}r} \\ &= \frac{1+\beta}{2}\, A_0(\mathbf{y}+r\boldsymbol{\omega}) + \frac{1-\beta}{2}\, A_0(\mathbf{y}-r\boldsymbol{\omega}) \,. \end{aligned} \tag{12.4}$$

Note that positive energy states are translated with the asymptotic velocity $\boldsymbol{\omega} = c\boldsymbol{\omega}$, and negative energy states with $-c\boldsymbol{\omega}$. Combining the three strong limits yields

$$\begin{aligned} \underset{\bar{p}\to\infty}{\text{w-lim}}\ \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\, S\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} &= \left(\exp\left\{ i \int_0^{\infty} dr\, W(r) \right\} \right)^* \mathbf{1}\ \exp\left\{ i \int_0^{-\infty} dr\, W(r) \right\} \\ &= \exp\left\{ -i \int_{-\infty}^{\infty} dr\, W(r) \right\}. \end{aligned}$$

Now both $\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} S\,\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}$ and the weak limit are unitary, thus strong convergence is established. We employ the equation (12.4) for $W(r)$ and replace $-r$ by r in the integral of the second term to obtain (12.3).

The exponent in (12.3) contains the X-ray transform of the electrostatic potential $X(\mathbf{y},\boldsymbol{\omega}) = \int_{-\infty}^{\infty} dr\, A_0(\mathbf{y}+r\boldsymbol{\omega})$. X is continuous and vanishes for $|\mathbf{y}| \to \infty$ orthogonal to $\boldsymbol{\omega}$, thus it can be obtained uniquely from its exponential, and the potential A_0 is recovered as explained in Section 10.

Lemma 12.2 *Under the assumptions of Theorem 12.1 we have the limit*

$$\operatorname*{s-lim}_{\bar{p}\to\infty} \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} J^{-1}\Omega_\pm \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \exp\left\{ i\int_0^{\pm\infty} dr\, W(r)\right\}\,, \tag{12.5}$$

where $W(r)$ is given by (12.4).

To prove (12.5), we show first that it is sufficient to consider a finite time interval. We employ the dense subspace $\mathcal{D} := \{\Psi \in \mathcal{H}_0 \,|\, \hat{\psi} \in C_0^\infty(\mathbb{R}^\nu)\}$. For $\Psi \in \mathcal{D}$, we have the Bochner integral

$$\begin{aligned} &\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}(J^{-1}\Omega_\pm)\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi - \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} J^{-1}\mathrm{e}^{iH_1 t} J\,\mathrm{e}^{-iH_0 t}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi \\ =\;& i\int_t^{\pm\infty} ds\,\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} J^{-1}\mathrm{e}^{iH_1 s}(H_1 J - JH_0)\,\mathrm{e}^{-iH_0 s}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi\,, \end{aligned} \tag{12.6}$$

and the integrand is bounded by an integrable function $h(s)$, which is independent of $\bar{p} \ge \bar{p}_0$. Setting

$$V := J^{-1}H_1 J - H_0 = \begin{pmatrix} 0 & 0 \\ iA_0^2(\mathbf{y})\,\frac{1}{\sqrt{\mathbf{p}^2+m^2}} & 2A_0(\mathbf{y}) \end{pmatrix}\,, \tag{12.7}$$

$h(s)$ is obtained from the decomposition

$$\begin{aligned} &\left\| V\,\mathrm{e}^{-iH_0 s}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi \right\| \\ \le\;& \left\| V\,F(|\mathbf{y}| \le \tfrac{s}{2})\,\mathrm{e}^{-iH_0 s}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi\right\| + \left\| V\,F(|\mathbf{y}| \ge \tfrac{s}{2})\,\mathrm{e}^{-iH_0 s}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi\right\| \\ \le\;& \left\| V\right\|\cdot\left\| F(|\mathbf{y}| \le \tfrac{s}{2})\,\mathrm{e}^{-iH_0 s}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi\right\| + \left\| V\,F(|\mathbf{y}| \ge \tfrac{s}{2})\right\|\cdot\left\|\Psi\right\|\,, \end{aligned}$$

where $F(\dots)$ is the multiplication with the characteristic function of the indicated region. Now the first term is bounded by $\dfrac{\mathrm{const}}{(1+s)^2}$ from a non-propagation property analogous to (3.10), and the second term is integrable by the decay properties of A_0. $\dfrac{s}{2}$ should be read as $\dfrac{cs}{2}$, where c is the velocity of light: For large $\bar{p}$ the velocity support of $\Psi_{\bar{\mathbf{p}}}$ is contained in $|\mathbf{v}| > c/2$. Now the LHS of (12.6) is bounded by $\pm\int_t^{\pm\infty} ds\,h(s)$ uniformly for $\bar{p} \ge \bar{p}_0$, and by an $\varepsilon/3$-trick we may interchange the limits $t \to \pm\infty$ and $\bar{p} \to \infty$. Thus it is sufficient to show

$$\lim_{\bar{p}\to\infty} \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} J^{-1}\mathrm{e}^{iH_1 t} J\,\mathrm{e}^{-iH_0 t}\mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}}\,\Psi = \exp\left\{ i\int_0^t dr\, W(r)\right\}\Psi\,. \tag{12.8}$$

We employ the Dyson-expansion

$$\begin{aligned} &\mathrm{e}^{-i\bar{p}\mathbf{y}} J^{-1} \mathrm{e}^{iH_1 t} J\, \mathrm{e}^{-iH_0 t} \mathrm{e}^{i\bar{p}\mathbf{y}}\, \Psi \\ = &\sum_{n=0}^{\infty} i^n \int_0^t dt_n \int_{t_n}^t dt_{n-1} \ldots \int_{t_3}^t dt_2 \int_{t_2}^t dt_1\, \mathrm{e}^{-i\bar{p}\mathbf{y}} V(t_n)\, V(t_{n-1}) \ldots V(t_2)\, V(t_1)\, \mathrm{e}^{i\bar{p}\mathbf{y}}\, \Psi \end{aligned}$$

with $V(t) := \mathrm{e}^{iH_0 t} V\, \mathrm{e}^{-iH_0 t}$, where V is given by (12.7). The n-th term is bounded by $(|t|\, \|V\|_{\mathcal{L}(\mathcal{H}_0)})^n \|\Psi\|/n!$ independently of $\bar{p}$, thus the limit $\bar{p} \to \infty$ can be taken term-wise. Now (12.8) follows from

$$\begin{aligned} &\lim_{\bar{p}\to\infty} \int_0^t dt_n \int_{t_n}^t dt_{n-1} \ldots \int_{t_3}^t dt_2 \int_{t_2}^t dt_1\, \mathrm{e}^{-i\bar{p}\mathbf{y}} V(t_n)\, V(t_{n-1}) \ldots V(t_2)\, V(t_1)\, \mathrm{e}^{i\bar{p}\mathbf{y}}\, \Psi \\ = &\int_0^t dt_n \int_{t_n}^t dt_{n-1} \ldots \int_{t_3}^t dt_2 \int_{t_2}^t dt_1\, W(t_n)\, W(t_{n-1}) \ldots W(t_2)\, W(t_1)\, \Psi \\ = &\frac{1}{n!} \int_0^t dt_n \int_0^t dt_{n-1} \ldots \int_0^t dt_2 \int_0^t dt_1\, W(t_n)\, W(t_{n-1}) \ldots W(t_2)\, W(t_1)\, \Psi \\ = &\frac{1}{n!} \Big(\int_0^t dr\, W(r) \Big)^n \Psi\ . \end{aligned}$$

This is proved in [16], and here we shall sketch the proof for $n = 1$:

$$\lim_{\bar{p}\to\infty} \int_0^t dr\, \mathrm{e}^{-i\bar{p}\mathbf{y}} V(r)\, \mathrm{e}^{i\bar{p}\mathbf{y}}\, \Psi = \int_0^t dr\, W(r)\, \Psi\ . \tag{12.9}$$

The integrand can be written as a product
$\Big(\mathrm{e}^{-i\bar{p}\mathbf{y}} \mathrm{e}^{iH_0 r} \mathrm{e}^{i\bar{p}\mathbf{y}}\Big) \Big(\mathrm{e}^{-i\bar{p}\mathbf{y}} V\, \mathrm{e}^{i\bar{p}\mathbf{y}}\Big) \Big(\mathrm{e}^{-i\bar{p}\mathbf{y}} \mathrm{e}^{-iH_0 r} \mathrm{e}^{i\bar{p}\mathbf{y}}\Big)\, \Psi$ with
$\mathrm{e}^{-i\bar{p}\mathbf{y}} \mathrm{e}^{\pm iH_0 r} \mathrm{e}^{i\bar{p}\mathbf{y}} = \mathrm{e}^{\pm i\beta\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2 + m^2}\, r}$. We have
$\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2 + m^2} - (\bar{p} + \boldsymbol{\omega}\cdot\mathbf{p}) \to 0$ for $\bar{p} \to \infty$ and $\mathbf{p} \in \mathbb{R}^\nu$. Thus

$$\begin{aligned} &\Big(\mathrm{e}^{i\beta\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2 + m^2}\, t} - \mathrm{e}^{i\beta(\bar{p} + \boldsymbol{\omega}\cdot\mathbf{p})t}\Big)\, \Psi \\ = \ &\mathrm{e}^{i\beta(\bar{p} + \boldsymbol{\omega}\cdot\mathbf{p})t} \Big(\mathrm{e}^{i\beta(\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2 + m^2} - \bar{p} - \boldsymbol{\omega}\cdot\mathbf{p})t} - 1\Big)\, \Psi \to 0 \end{aligned} \tag{12.10}$$

for all $\Psi \in \mathcal{H}_0$ (by the dominated convergence theorem applied to $\int d^\nu p\, |\ldots \hat{\psi}(\mathbf{p})|^2$). This equation shows that in the high-energy limit, the spreading of a wave packet is negligible compared to the translation. We have

$$\mathrm{e}^{-i\bar{p}\mathbf{y}} V\, \mathrm{e}^{i\bar{p}\mathbf{y}} = \begin{pmatrix} 0 & 0 \\ iA_0^2 \frac{1}{\sqrt{(\mathbf{p}+\bar{\mathbf{p}})^2 + m^2}} & 2A_0(\mathbf{y}) \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 2A_0(\mathbf{y}) \end{pmatrix}, \tag{12.11}$$

and thus the integral on the LHS of (12.9) is asymptotically

$$\begin{aligned} &\int_0^t dr\, \mathrm{e}^{i\beta(\bar{p} + \boldsymbol{\omega}\cdot\mathbf{p})r} \begin{pmatrix} 0 & 0 \\ 0 & 2A_0 \end{pmatrix} \mathrm{e}^{-i\beta(\bar{p} + \boldsymbol{\omega}\cdot\mathbf{p})r}\, \Psi \\ = &\int_0^t dr\, \mathrm{e}^{i\beta 2\bar{p}r}\, \mathrm{e}^{i\beta\boldsymbol{\omega}\cdot\mathbf{p}r} \begin{pmatrix} -A_0 & 0 \\ 0 & A_0 \end{pmatrix} \mathrm{e}^{-i\beta\boldsymbol{\omega}\cdot\mathbf{p}r}\, \Psi \\ + &\int_0^t dr\, \mathrm{e}^{i\beta\boldsymbol{\omega}\cdot\mathbf{p}r} \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix} \mathrm{e}^{-i\beta\boldsymbol{\omega}\cdot\mathbf{p}r}\, \Psi\ , \end{aligned}$$

since the second matrix commutes with β, while the first is anti-commuting with β. Now the Riemann-Lebesgue Lemma is valid for the Bochner integral, thus the integral of the first term vanishes for $\bar{p} \to \infty$, due to cancellations by rapid oscillations. The second integral yields the RHS of (12.9).

This completes our sketch of the proof of Lemma 12.2, and thus of Theorem 12.1. It can be generalized to include an electromagnetic field (A_0, $\mathbf{A}$): Then the Klein-Gordon equation reads

$$\ddot{u} + i2A_0\,\dot{u} + \Big[(\mathbf{p}-\mathbf{A})^2 + m^2 - A_0^2\Big]u = 0\,,$$

and the operators and Hilbert spaces are defined in a similar way, where B_1^2 is changed to $B_1^2 = (\mathbf{p} - \mathbf{A}(\mathbf{x}))^2 + m^2 - A_0(\mathbf{x})^2$. We have

Theorem 12.3 *Suppose that $\nu \in \mathbb{N}$, $m > 0$ and S is the scattering operator for a Klein-Gordon particle of mass m in an electromagnetic field $(A_0,\, \mathbf{A})$, which is bounded and decays integrably: For $A \in \{A_0,\, \mathbf{A},\, \operatorname{div}\mathbf{A}\}$ we have $A \in L^\infty(\mathbb{R}^\nu)$ and $\int_0^\infty dR\, \|\chi(|\mathbf{x}| > R)\, A(\mathbf{x})\|_\infty < \infty$. Moreover, we make the Kato-Rellich assumption $\|(B_1^2 - B_0^2)\Psi\| \le a\|B_0^2\Psi\|$ with $a < 1$. Then the high-energy asymptotics of S are given by*

$$\operatorname*{s-lim}_{\bar{p}\to\infty} \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} S\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \exp\left\{-i\int_{-\infty}^{\infty} dr\, \begin{pmatrix} A_0 & -i\boldsymbol{\omega}\cdot\mathbf{A} \\ i\boldsymbol{\omega}\cdot\mathbf{A} & A_0 \end{pmatrix} (\mathbf{y}+r\boldsymbol{\omega})\right\}, \qquad (12.12)$$

where $\bar{\mathbf{p}} = \bar{p}\boldsymbol{\omega}$ with $\boldsymbol{\omega} \in S^{\nu-1}$. Denoting the restriction of S onto the subspace of positive/negative energy by $S_\pm$, we obtain

$$\operatorname*{s-lim}_{\bar{p}\to\infty} \mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} S_\pm\, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} = \exp\left\{-i\int_{-\infty}^{\infty} dr\, \Big(A_0 \mp \boldsymbol{\omega}\mathbf{A}\Big)(\mathbf{y}+r\boldsymbol{\omega})\right\}. \qquad (12.13)$$

If $\nu \ge 2$ and A_0, $\mathbf{A}$ are continuous, then A_0 and $\mathbf{B} = \operatorname{rot}\mathbf{A} \in \mathcal{S}'$ can be reconstructed uniquely from S or S_+ (we need an additional technical assumption on $\mathbf{A}$, e.g. $\mathbf{A} \in L^2$ is sufficient).

This theorem was announced in [15], and a complete proof will be given in [16]. Note that equation (12.13) is the same for the Dirac equation, which was treated in [15] by a similar approach, where the decay assumptions on the potentials are the same, but the Kato-Rellich condition looks less restrictive. The inverse scattering problem for the Dirac equation was solved by stationary methods in [13] and [12] under stronger assumptions on the potentials, and the geometrical, time-dependent method of [15] was extended in [14] to cover time-dependent electromagnetic fields. In [1], the inverse scattering problem for the Schrödinger equation with electric and magnetic potentials was solved with the geometrical method.

Acoustic waves in an inhomogeneous medium are described by $\ddot{u} = c^2\rho\,\nabla\frac{1}{\rho}\nabla u$. If $c(\mathbf{x}) \to 1$ and $\rho(\mathbf{x}) \to 1$ for $|\mathbf{x}| \to \infty$ suitably, there is a scattering theory with the corresponding free equation given by $\ddot{u} = \Delta u$. The high-energy asymptotics will be described by the eikonal equation and are not easily obtained from our time-dependent approach. If we consider the special case of $c \equiv 1$, thus $\ddot{u} = \Delta u - \frac{1}{\rho}\nabla\rho \cdot \nabla u$, the high-energy limit is calculated in the same way as for the Klein-Gordon equation with magnetic field, and $\Big(\frac{1}{\rho}\boldsymbol{\omega}\cdot\nabla\rho\Big)(\mathbf{y}+\boldsymbol{\omega}r) = \frac{\partial}{\partial r}\log\rho(\mathbf{y}+\boldsymbol{\omega}r)$ yields $\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}}\Omega_\pm \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} \to \rho^{1/2}$.

Together with $\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} g \, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} \to \rho^{-1}$, the limit $\mathrm{e}^{-i\bar{\mathbf{p}}\mathbf{y}} S \, \mathrm{e}^{i\bar{\mathbf{p}}\mathbf{y}} \to 1$ is obtained. See [16] for details.

The books [17] and [19] give the background on mathematical scattering theory for Schrödinger and Dirac operators. The paper [5] introduces the method in an elementary way for non-specialists and gives many references.

The papers [1], [2], [5]-[9], and [15] (preprint versions) can be downloaded via our homepages `http://www.iram.rwth-aachen.de/~enss` and `.../~jung` or by FTP from `ftp.iram.rwth-aachen.de/pub/papers/...` or from `mp_arc`.

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